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PROCEEDINGS OF THE FOURTH ORDNANCE CONFERENCE ON OPERATIONS RESEARCH

OFFICE OF ORDNANCE RESEARCH, U.S. ARMY
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Office of Ordnance Research

Report No. 59-3

PROCEEDINGS OF THE FOURTH ORDNANCE CONFERENCE ON OPERATIONS RESEARCH

Held at

Headquarters, U.S. Ordnance Missile Command
1-3 April 1959

Office of Ordnance Research
Ordnance Corps, U.S. Army
Box CM, Duke Station
Durham, North Carolina

TABLE OF CONTENTS

*This paper was presented at the conference. It does not appear in these Proceedings.

**This paper can be found in a classified security information (SECRET) appendix of this Report.

TABLE OF CONTENTS (Cont'd)

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FOREWORD

The present series of conferences was initiated on 14 May 1954 at Frankford Arsenal in order to disseminate information on the methods and new developments in the field of operations research to a large number of government personnel. On 20 May 1955 the Diamond Ordnance Fuze Laboratories served as host to the second conference, and the third meeting was held 24-25 May 1956 at Headquarters, Ordnance Weapons Command. Late in 1958 Oscar M. Wells expressed the need for more symposia in this area; and thanks to the invitation issued by Emory L. Atkins, the Fourth Ordnance Conference on Operations Research was held at Headquarters, U. S. Ordnance Missile Command on 1-3 April 1959.

Colonel George F. Leist, Ordnance Corps Commanding Officer of the Office of Ordnance Research, served as chairman of the first General Session at which Brigadier General John G. Shinkle, Commander, Army Rocket and Guided Missile Agency, gave the Welcoming Remarks. He also presented a brief history of Redstone Arsenal. The initial invited address, by Gayle W. McElrath of the University of Minnesota, emphasized the need for careful designing and proper planning of experiments. The second invited paper, by J. E. Hoagbin, The Willow Run Laboratories of the University of Michigan, presented a summation of the development of a mathematical model of a queuing and scheduling problem. Dr. John W. Coy, White Sands Missile Range, served as chairman of the first Technical Session. In this group were four papers covering such topics as missile system testing, war games, discounted least squares, and waiting line theory. Technical Sessions II and IV had Emory L. Atkins and Robert R. Orr from the Army Ordnance Missile Command as their respective chairmen. Both of these meetings carried security classifications of SECRET. Mr. Oscar M. Wells presided over Technical Session III, in which four papers treated demand rates, tolerance limit tests, production development, and applications of linear programming. The final phase of the conference was chairmaned by Ralph Brown of Frankford Arsenal. In this General Session O. M. Wells discussed various types of arsenal problems that have been found amenable to operations research techniques. Lt. Colonel Harold W. Rice, Research Section, Plans Division, Deputy Chief of Staff for Logistics, gave an invited address entitled Army Logistics Research. The closing address of the meeting was delivered by the invited speaker, Herbert P. Galliher, of the Massachusetts Institute of Technology. Dr. Galliher spoke on Future Operations Research Problems in Ordnance Logistics.

These Proceedings contain thirteen of the nineteen papers that were presented at the Fourth Ordnance Conference on Operations Research. An appendix to this technical manual carrying a security classification of SECRET is being issued. This appendix will contain papers by J. Duncan Love, Eugene P. Visco, John P. Young, and Richard Zimmerman, all of the Operations Research Office.

PROGRAM

FOURTH ORDNANCE CONFERENCE ON OPERATIONS RESEARCH
Headquarters, U. S. Ordnance Missile Command
1 - 3 April 1959

1 April

MORNING SESSION:	0900 - 1200 - Rocket Auditorium, Building 7120 Chairman: Colonel George F. Leist, Ordnance Corps Commanding Officer of the Office of Ordnance Research
0915 - 0945	Introductory Remarks: Brigadier General John G. Shinkle, Commander, Army Rocket and Guided Missile Agency
0945 - 1045	Design of Experiments G. W. McElrath, Industrial Engineering Division, University of Minnesota
1045 - 1100	Break
1100 - 1200	Mathematical Models and the Utilization of these Models J. E. Hoagbin, The Willow Run Laboratories of the University of Michigan
LUNCH	1200 - 1315
TECHNICAL SESSION I:	1315 - 1615 - Rocket Auditorium, Bldg 7120 Chairman: John W. Coy, Scientific Advisory Ordnance Mission, White Sands Missile Range
1315 - 1400	Computer Played War Games John J. Dunne, Weapon Systems Laboratory, Ballistic Research Laboratories, Aberdeen Proving Ground
1400 - 1445	Aspects of Operations Research in Missile System Testing Forrest L. Staffanson, Ordnance Mission, White Sands Missile Range
1445 - 1500	Break
1500 - 1545	Discounted Least Squares R. J. Duffin and Th. W. Schmidt, Office of Ordnance Research and Duke University
1545 - 1615	A Practical Application of the Waiting Line Theory for the Optimum Allocation of Machine Repair Operators T. Moundalexis, Arsenal Operations Division, Picatinny Arsenal

TECHNICAL SESSION II: 1315 - 1600 - Room A-100, Bldg. 4488
Chairman: E. L. Atkins, Chief, Operations Research Division, AC/S, Military Applications & Training, AOMC

Security Classification - SECRET

1315 - 1405 Utility Testing of Prototypes
William C. Pettijohn, Operations Research Office, JHU

1405 - 1455 Casualties from a Surprise Chemical Attack
Eugene P. Visco, Operations Research Office, JHU

1455 - 1515 Break

1515 - 1600 An Analytical Method for Relating Anti-Tank Weapon Characteristics to User Needs
Joseph A. Bruner and John P. Young, Operations Research Office, Johns Hopkins University

1830 Social Hour and Dinner - Officers' Open Mess, Bldg. 111

2 April 1959

TECHNICAL SESSION III: 0900 - 1200 - Rocket Auditorium, Bldg. 7120
Chairman: Oscar M. Wells, Operations Research Office, Headquarters, Ordnance Weapons Command

0900 - 0940 Linear Programming Applied to a Foundry Cost Problem
Gideon I. Gartner, General Thomas J. Rodman Laboratory, Watertown Arsenal

0940 - 1020 An Optimal Method for Production Development (Evaluation, Selection and Programming)
Sidney Sobelman, Program Coordination Office, Picatinny Arsenal

1020 - 1035 Break

1035 - 1115 The Joint Frequency Function Approach to Tolerance Limit Tests
Joseph P. Fearey and Arthur J. Stasney, Jet Propulsion Laboratory

1115 - 1200 Analysis of Demand Rates
Louis F. Nanni, Statistical Section, Raritan Arsenal

TECHNICAL SESSION IV: 0900 - 1200 - Room A-100, Bldg. 4488
Chairman: Robert R. Orr, Operations Research Division, AC/S Military Application & Training, AOMC

Security Classification. The first two papers in this session are classified SECRET and the last two carry a security classification of CONFIDENTIAL.

0900 - 0940	Historical Analysis of Artillery Fire J. Duncan Love, Operations Research Office, JHU
0940 - 1020	FAME: A theatre War Game Richard Zimmerman, Operations Research Office, JHU
1020 - 1040	Break
1040 - 1120	A Game-Theoretic Analysis of a Small Tank Unit in a New-Type Mine Field Raymond H. Burros, Combat Operations Research Group, Combat Development Section, Hq., USCONARC
1120 - 1200	A Computer Simulation of Small Combat Actions John C. Flannagan, Combat Operations Research Group, Combat Development Section, Hq., USCONARC

LUNCH: 1200 - 1315

GENERAL SESSION: 1315 - 1600 - Rocket Auditorium, Bldg. 7120
Chairman: Ralph Brown, Frankford Arsenal

1315 - 1400	Operations Research at the Ordnance Weapons Command Oscar M. Wells, Operations Research Office, Headquarters, Ordnance Weapons Command
1400 - 1445	Army Operations Research Lt. Colonel Harold W. Rice, Research Section, Plans Division, Deputy Chief of Staff for Logistics
1445 - 1500	Break
1500 - 1600	Future Operations Research Problems in Ordnance Logistics Herbert P. Galliher, Massachusetts Institute of Technology

3 April 1959

0830 - 1200	TOUR

SOME ASPECTS OF EXPERIMENTAL DESIGN IN OPERATIONS RESEARCH

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INTRODUCTION. Basically the ideas of designed experiments are very fundamental and have numerous applications in the various areas of operations research. It becomes apparent that the usefulness of the analysis of data depends greatly upon a systematic planning of the data in addition to the necessary assumptions of independence and distribution. The intent of this paper is two-fold: first, to acquaint the reader with concepts and assumptions ingredient in statistical designs; second, to present examples which illustrate some of the basic properties of designed experiments. The material presented is not intended to be complete in nature. It is not attempting to suggest many experimental designs together with many applications nor is it attempting to exhaust all the possibilities of designed experiments. It must follow, then, that the dominant mission of this paper is to establish an appreciation and dynamic interest strong enough to motivate the reader to further study designed experiments together with the appropriate statistical techniques for intelligent analysis of data when the need for such refinement exists.

In operations research a person may be called upon to design laboratory or field experiments to test differences between alternate methods, processes, or products. He may also analyze operational problems by taking samples to give quick estimates of short-run processes or performance forecasts. In addition he may be called on to spot trouble and appraise results. In all of these cases one is faced with the problem of decision making and estimation in the face of random variability.

It should be observed that the field of statistical decision making and estimation is relatively new but one that is gaining support at a rapid rate. To some, the pressure of work and what seems to be an unreasonable amount of technical terminology combine to create the impression that too much time and effort are necessary to understand a sufficient amount of the design and analysis of experiments. However, a sincere effort to learn and to use these techniques is early rewarded by finding that the basic fundamental ideas are natural, simple, and extremely useful.

GENERAL CONSIDERATIONS. A classical accepted method of experimentation is to answer a single question about a single variable. There is an attempt to control all other "factors" except the one of interest. Many times the planning consists of trying first one idea, and if the one idea does not produce "satisfactory results" then try another idea. However, data based on this type of experimentation often consists of a number of disjoint pieces of information that cannot be easily put together. It is possible to design experiments such that all combinations of the different factor "levels" can be considered in order that their "individual effects" together with their "interactions" may be studied.

When the term DESIGN OF EXPERIMENT is used, a reference is made to a plan of choosing a sample. For fear that the reader think only in terms of sampling a process that exists, let the author hasten to assure him that many "designed experiments" are called upon by means of planned sampling to generate information about a process that does not actually exist. The objective is to discover the properties of the process if the process did exist. Much of the exploratory operations research is carried on in an environment where the process does not exist. Of course, a second situation exists when there is actually a real process and the objective is to determine certain measurable characteristics relative to this process. Again there are special "designed experiments" which can give more complete and precise information about this process than just haphazard sampling. The analysis of the data in the designed experiment forces one to use the concepts of statistical significance and estimation which in turn force one to think in terms of decisions and estimations with calculated risks.

Again it should be emphasized that statistical decision making and estimation via designed experiments is done in the face of random variability. This philosophy is fundamental to many areas of operations research.

Anyone can try to make sense out of a set of observational or experimental data. However, the ability to scrutinize, to look beneath the surface and to discern relationships from given data is aided by scientific methods of collecting, analyzing, and interpreting data. The modern methods of designed experiment and analysis deal with such questions as:

1. What factors are considered important and of interest?

2. What type of designed experiment should be used?

3. How shall a program of obtaining the data be planned?

4. How shall the data be analyzed?

5. What conclusions are we entitled to draw from the data?

6. How shall the experimental error be estimated and how reliable are the conclusions?

To answer these questions completely in research engineering, or production, is an impossibility. However, we may gain insight of the basic philosophy.

The first three questions in the above list are often slighted or are not considered before the data is gathered. Too many times a project is started, data is obtained, and a post-mortem analysis of the data is held in a faint hope of finding something "significant." The criterion of significance in this situation often depends upon personal judgment. Before an effective program of obtaining data can be planned a STATEMENT OF THE PROBLEM is indeed necessary. This statement might very well take the form of a list of questions concerning the factors of interest. When this list of questions has been generated, perhaps after sufficient exploration, a most important prerequisite of a designed experiment has been satisfied.

The problems associated with experimental operations research are basically problems that demand an analysis of the variation of the data in an attempt (a) to determine which variables are significant and (b) to determine interval estimates of the effects of the variables considered. To this end, statistical methods may be used to analyze the data from the designed experiment to calculate the total amount of variation which is sub-divided into categories traceable to definite sources. The relative sizes of these key categories of variation are compared with the size of the "experimental error." The significance of the effects of the variables together with the interval estimates can be judged as an outcome of these comparisons. The sensitivity of the experiment depends upon the magnitude of this experimental error; this error becomes the yardstick of the experiment. The size of the error determines the ability to pick up effects that are significant and thereby dictates a "level of comparison." A large experimental error might very well mask the effects of important variables which are present. The ideal situation would dictate a minimum experimental error; however, we are usually forced to indicate an "economic" minimum. Our problem in planning a program for obtaining data is two-fold: namely, that of estimating the experimental error, and that of determining an economic experimental error (yardstick). A pre-planned experiment provides at least three ways of controlling the experimental error. They are as follows:

1. The degree of refinement of the measurement techniques.

2. The selection of the number of replications.

3. The choice of the DESIGN OF THE EXPERIMENT.

<u>SOME BASIC DESIGNS</u>. The SUBJECT of a designed experiment is the dependent measureable characteristic. Other variables which appear to affect the subject are called FACTORS. The factors of an experiment may be varied deliberately from trial to trial. The relation of subject to factors can be measured by proper planning of the given factors controlled at a given number of LEVELS. A TREATMENT is that combination of levels of all of the factors which are considered in a given trial. The numerical value corresponding to a given trial is the OBSERVATION or RESPONSE.

Consider the following introductory remarks that suggest the evolution of a designed experiment which allows an unbiased estimate of the experimental error and at the same time introduces a method to minimize the error by restricting the randomization.

The following problem is used to present some very important fundamentals which should be a part of operations research planning of experiments. The problem is to test for a differential effect of materials from four vendors. The material is aluminum clad and is to be spot welded. The experiment is motivated by an interest to select the material which responds in the best manner to spot welding at given specifications. However, it is believed that specific types of spot welders have different characteristics and that these characteristics become important when the engineer is faced with a recommendation for routing the material in a given process. Consider the subject to be tensile strength and the factor to be materials. We shall label the materials as A, B, C, and D. Let us

consider a very practical situation in which the aluminum clad material is welded by four different machines, 1, 2, 3, and 4. Although the machines are set to spot weld to specification, it will be necessary to consider these machines as another factor.

It is obviously faulty experimental design to lay out the experiment as shown schematically in Figure 1. If a difference appears we do not know whether the difference is due to machines or materials. The materials effect is completely "confounded" with the machines effect and the two are not separable.

Machine 1	Machine 2	Machine 3	Machine 4
Material A	Material B	Material C	Material D
Material A	Material B	Material C	Material D
Material A	Material B	Material C	Material D
Material A	Material B	Material C	Material D

Figure 1.

A suggested improvement on the above procedure is to "completely randomize" the materials among the machines. For example, we could identify A, B, C, D four times by 16 random numbers. Each machine would spot weld the four materials as they occurred by this "complete randomization." See Figure 2.

Machine 1	Machine 2	Machine 3	Machine 4
Material A	Material B	Material C	Material B
Material C	Material A	Material D	Material B
Material C	Material D	Material B	Material A
Material A	Material C	Material D	Material D

Figure 2.

The design suggested in Figure 2 is statistically sound. We can take the averages of the four results for each material to estimate their process means. However, in a completely randomized design the experimental error may be inflated because it includes the differences of the machines. Thus the experiment might not be as sensitive as possible.

Another suggestion, in the face of the above discussion, is to have each machine process each material once as indicated in Figure 3.

Machine 1	Machine 2	Machine 3	Machine 4
Material A	Material A	Material A	Material A
Material B	Material B	Material B	Material B
Material C	Material C	Material C	Material C
Material D	Material D	Material D	Material D

Figure 3.

In this situation the differences in materials are completely independent of machines. However, in order not to introduce any possible bias into the experiment because of the identical ordering for each machine, a further refinement is suggested. That is, to randomize the order of materials to be spot welded for each machine as in Figure 4.

Machine 1	Machine 2	Machine 3	Machine 4
Material A	Material C	Material B	Material D
Material C	Material B	Material C	Material B
Material D	Material D	Material D	Material A
Material B	Material A	Material A	Material D

Figure 4.

The above basic design is one of the simplest and perhaps most frequently used experimental designs used in operations research, and is called the "randomized block" design. A conventional notation would label machines as "blocks" and materials as "treatments." The blocks are so chosen that units in a block have some common feature or background likely to make them uniform. If possible, every unit in a block receives identical handling throughout the experiment except for one factor: the experimental treatment (materials, in this example). Thus we identify this experiment as single grouping to reduce the experimental error. The statistical technique applicable to analyze the data obtained from the randomized block design is the "analysis of variance." By using the analysis of variance we can divide the total variability into three "sources of variation"; namely, materials, machines, and experimental error. Thus, material effect can be compared with the experimental error to determine the materials significance, independent of whether there is a significant variation from machine to machine.

Note that the possibility for gain in sensitivity in a randomized block design arises from further restricting the randomization which results in the grouping that is employed in the experiment. The principle is to use groupings within which the product is likely to be less variable than the product as a whole. This is an analog to rational subgroups which is used so frequently in analysis of process variation by statistical quality control techniques.

Features of randomized blocks: *

1. One category of factors can be permitted to vary from one block to another, yet the main component of this variation becomes excluded from the residual error of the experiment.

2. Any number of treatments that is desired can be used, so long as each block has the same set of treatments.

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* See <u>Quality Control Handbook</u>, J. M. Juran, McGraw-Hill Book Co., Inc., 1951.

3. As many repeat tests (called "replications") as are wanted can be obtained by adding blocks.

4. Analysis of remaining data can still be made even if some failure in process or test eliminates an entire block.

MACHINES

	1	2	3	4
I	Material B	Material A	Material C	Material D
II	Material D	Material C	Material A	Material B
III	Material A	Material D	Material B	Material C
IV	Material C	Material B	Material D	Material A

Figure 5.

A natural extension of the concept of single grouping to reduce errors is to suggest a double grouping. This is possible by an additional restriction of the randomization of units to be tested.

For example, it is well known that the tensile strength of the spot welds can be influenced by the techniques of the operators I, II, III, and IV, on the machines. By advanced planning the design with double grouping requires that each material is processed just once respectively with each machine and operator combination. It follows, then, that the number of machines, materials and operators must all be the same. An experimental design of this type is called a Latin - square experiment. The scheme is given in Figure 5.

As before, the observed variation in tensile strength can be classified. If the subject of the experiment satisfies certain basic assumptions to the analysis of variance, we can measure the significance of the three sources of variation; namely, the materials, the machines, and the operators. The criterion of significance again is determined by comparison with the experimental error.

However, there exists a restricting assumption in the analysis of the Latin - square data. That is, this design will not measure "interaction." Interaction is basically an additional response in one factor due to the presence of another factor at a specific level. The assumption of no interaction can either be tacit or explicitly determined. (Much of our classical experimentation incorrectly assumes that the factors are independent and that the interaction does not exist.)

An elementary experiment which has the interaction effect designed into it is a two factor design with repeated measurements. If there is a possibility for significant interaction, the estimate of the experimental error for use in investigating significant effects of the factors is obtained from measurements repeated under the same conditions. This replication will allow us to analyze the data more fully.

Suppose that in a study of tensile strength, the two factors are materials and machines, and that a sample of three coupons of the same aluminum clad material are spot welded by the same machine. The design may take the appearance of Figure 6.

	MACHINES 1	2	3
A	Sample 1A Sample 1A Sample 1A	Sample 2A Sample 2A Sample 2A	Sample 3A Sample 3A Sample 3A
B	Sample 1B Sample 1B Sample 1B	Sample 2B Sample 2B Sample 2B	Sample 3B Sample 3B Sample 3B
C	Sample 1C Sample 1C Sample 1C	Sample 2C Sample 2C Sample 2C	Sample 3C Sample 3C Sample 3C
D	Sample 1D Sample 1D Sample 1D	Sample 2D Sample 2D Sample 2D	Sample 3D Sample 3D Sample 3D

Figure 6.

To analyze the data of this experiment we can again use the statistical technique called the analysis of variance. We divide the source of total variation into four different categories; namely, materials, machines, interaction between materials and machines, and experimental error. Thus we have a yardstick in terms of the experimental error to determine the significance of the factors involved. At least by this designed experiment, together with its analysis, we can determine statistically significant factors in an objective manner and can determine the risks of wrong decisions. The wrong decisions are of two types; namely, the decision that there is no significant difference when the difference actually exists or the decision that there is a significant difference when the difference does not exist.

SUMMARY. Previously we asked four questions concerning scientific methods of handling data. The emphasis of this paper was concerned with basic concepts ingredient in the design of the experiments.

To repeat, the classical accepted method of experimentation has been to answer a single question about a single variable. That is, there was an attempt to control all other factors except the one of interest. However, it is often found that data based on the control of all factors, except the one under investigation, consists of a number of disjoint pieces of information that cannot be easily put together.

By method of designed experiments, plans are available which consider the investigation of the effects of a number of independent factors at the same time. The author wishes to emphasize that the desired information be planned and designed into the experiment before the actual data is obtained.

There are several advantages to be gained by the operations researcher who thinks in terms of designed experiments. Three important advantages may be summarized as follows:

1. Greater efficiency - therefore, greater economy of experimentation.

2. Results can be obtained which apply over a greater range of conditions.

3. Main effects together with their interactions can be estimated.

The inferences drawn from designed experiments are valid inferences containing a measure of uncertainty. However, these inferences are rigorous when made by statistical analysis, since they include a probability measure of the amount of uncertainty involved. This achievement makes it possible to determine the reliability of the process of acquiring new knowledge by observation. Thus it becomes possible to accomplish one important goal of operations research.

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MATHEMATICAL MODELS AND THE UTILIZATION OF THESE MODELS

J. E. Hoagbin
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It is frequently necessary to conduct experiments using a mathematical model of a system rather than the system itself. Some of the possible reasons for this are:

1. It is necessary to experiment with a system that has not yet been built, although intended properties of components are known.

2. It is impossible to assemble all components for experimentation, even though they have been built.

3. It is impossible or uneconomical to train operators in the use of many alternative rules of operation. The plan is to experiment with many rules of operation and then train operators to use the rules that have been found to be best.

4. Because of the many stochastic elements in the systems, it is necessary to conduct many experiments to estimate the mean and the variance of the property that is to be measured during the experimentation program. It is economically unsound to conduct many experiments with the system, or the system cannot be made available for long periods of experimentation.

5. It is impossible to conduct the experiments for moral reasons. For example, we cannot use real bullets in a maneuver or air exercise.

Frequently equations can be written describing system behavior. The equations represent a model of the systemand experimentation with the model consists of solving the equations for different values of parameters. Such mathematical models are well known and include the models that have grown out of linear programming, for example.

The rapid development of Monte Carlo techniques and the large scale digital computer have made possible the construction of a whole new class of model. It is this latter class of model that will be discussed in this talk.

DIGITAL COMPUTER MODELS.

Digital computer models can now be constructed to study problems that are difficult or impossible to study by the use of classical mathematical analysis for such reasons as:

1. Equations expressing system behavior cannot be solved because of discontinuities and non-linearities.

2. Knowledge of the transient behavior of the system is necessary, and existing methods of classical analysis give no information about transient behavior.

3. There are stochastic elements that cannot be handled by classical methods.

Many military systems involve the flow of information or material; and attention is drawn to capacities, accuracies, and time delays. Simulation of such a system requires the solution of many technical computations as well as the solution of a complex waiting-line problem.

A SIMPLE DEMONSTRATION OF THE USE OF MONTE CARLO TO STUDY A WAITING-LINE PROBLEM.

Consider a hypothetical operator who must process inputs. As new inputs arrive, the operator takes them in turn if he is free. If he is busy when an input arrives, the input must wait for service in a queue or waiting line. Occasionally the operator makes mistakes and produces scrap, rather than properly finished inputs.

The total time required for inputs to pass through this simple system, the time spent by inputs waiting for service, and the number of inputs that are properly processed in a given period depend on the following: (1) the distribution of arrivals or randomness of the arrivals and average rate of arrival, (2) the distribution of operator holding times or randomness of the time required for the operator to process an input and the average rate at which the operator can process inputs, and (3) the probability the operator will make a mistake and produce scrap rather than properly finished inputs.

The distribution of arrivals may be a function of parameters, but assume the distribution is that shown in Table I. Presentation of the histogram used to produce Table I in tabular form will be useful in the demonstration of sampling from distributions that will come later.

It is important to emphasize that the long and short intervals between inputs occur randomly and are not attributable to any known factors. Thus, just as any two-digit random number between 00 and 99 is equally likely to occur on a singly drawing, any of the time intervals can occur, although the occurrence of some are more probable than others.

Just as there is some randomness in the intervals between successive arrivals, there is randomness in the time required for the operator to process an input. When factors affecting the length of service times or holding times are known, the distribution of holding times can be made a function of these factors.

Table II shows the distribution of service times for the operator under consideration. Note that the distribution is a function of the length of the queue.

TABLE I, ASSUMED DISTRIBUTION OF ARRIVALS OF INPUTS TO A HYPOTHETICAL OPERATOR.

To use the table, draw a two-digit random number, find the row corresponding to this number in the first column, and read a time interval in the column to the right.

Random Number	Time Interval
01-02	3.0 minutes
03-05	2.5
06-07	2.0
08-11	1.75
12-15	1.5
16-22	1.25
23-29	1.0
30-41	.75
42-49	.5
50-57	.4
58-69	.3
70-79	.2
80-87	.10
88-94	.05
95-100*	.03

*Read 00 as 100.

TABLE I

TABLE II, HYPOTHETICAL DISTRIBUTION OF HOLDING TIMES FOR AN OPERATOR WHOSE HOLDING TIMES ARE A FUNCTION OF THE LENGTH OF THE QUEUE.

To use this table: draw a two-digit random number, look in the column corresponding to the length of the queue, and read the length of time required for the operator to perform his function in the column on the right, opposite the random number chosen.

Length of Queue				Time Required
0	1	2	3 or more	
			00-01	.3 minutes
		00-01	02-10	.4
	00-01	02-17	11-50	.5
00-01	02-10	18-45	51-80	.6
02-05	11-30	46-75	81-95	.7
06-20	31-55	76-90	96-98	.8
21-44	56-78	91-96	99	.9
41-60	79-88	97-99		1.0
61-80	89-96			1.1
81-90	97-98			1.2
91-95	99			1.3
96-98				1.4
99				1.5
				1.6
				1.7

TABLE II

TABLE III, HYPOTHETICAL PROBABILITY AN OPERATOR WILL PRODUCE SCRAP.

Length of Queue	Probability of Producing Scrap
0	.10
1	.15
2	.18
3 or more	.20

TABLE III

The probability an operator will produce scrap may also be a function of parameters. Table III shows the probability to be used in the demonstration and shows this probability is also a function of the length of the queue. Thus, when the operator works faster, he makes more mistakes.

To begin the demonstration, assume the operator has no inputs waiting, although the simulation could begin with some waiting line of interest. It is possible to determine in advance when inputs will arrive. The times of arrival of five inputs will be determined. Draw five two-digit random numbers and get 85, 32, 38, 64, and 97. Consulting Table I, the first input will arrive at time .1 minutes, the second .75 minutes thereafter, and so on. Call events of type E1 the event of the arrival of an input. We can construct the following table:

E1
.10
.85
1.6
1.9
1.93

The operator is free when the input arrives at time 0.1. To determine when he will finish working on the input, draw a random two-digit number and get 14. Consulting Table II, the operator will require .8 minutes to process the input. Thus, the operator will finish at time .9 which is .8 minutes after he started.

To determine whether the operator will produce scrap: Consulting Table III, if the number drawn is 01-10, the operator produced scrap. Draw the number 79. The operator will produce a properly finished input.

Call the event of type E2 the event that the operator has just finished processing an input. We now have the following table from which the first E1 was deleted:

E1	E2
.85	.9
1.6	
1.9	
1.93	

And just prior to time .85, the status of the system can be summarized as follows:

Inputs in the queue: None

Operator: Processing a properly finished input

Number of properly finished inputs: None

Number of scrapped inputs: None

Proceed to the next event that will occur. This is the event of type E1 at time .85 when a new input will arrive. Since the operator is still busy at this time, the new input must go into the queue; and the status of the system can be summarized as follows just before time .9:

Inputs in the queue: 1

Operator: Busy processing a good input

Number of properly finished inputs: None

Number of scrapped inputs: None

The events that will occur:

E1	E2
1.6	.9
1.9	
1.93	

Proceed immediately to time .9 to consider what happens when the operator finishes processing the input. At this time add 1 to the list of properly completed inputs, delete one from the queue, and determine when the operator will finish processing the second input.

Draw the random number 98 and consult Table II to learn the operator will require 1.4 minutes to process this input. Also draw the random number 08 and learn the input is destined to become scrap. Since the new E2 will come at time 2.3, the times recorded now are:

E1	E2
1.6	2.3
1.9	
1.93	

Thus, just prior to time 1.6, the status of the system is:

Inputs in the queue: None

Operator: Producing a scrap input

Number of properly finished inputs: 1

Number of scrapped inputs: None

If we proceed to the next event that will occur, we find the operator busy when the input arrives at time 1.6; hence, we add 1 to the queue. The next two events are also associated with arrivals of inputs; and just prior to time 2.3, there will be three inputs in the queue. At time 2.3, add 1 to to the list of scrapped inputs, subtract one from the queue, and again determine when the operator will finish processing the next input.

When the fifth input has been processed, no more times will remain in the list of events; and the simulation of the operator processing five inputs will be over.

The emphasis in the demonstration was on sampling from distributions and tracing of inputs. Nothing was said about recording information concerning, for example, the total time required to get inputs through the whole system. However. it is evident that one can record the time of arrival of inputs, the time service begins on each input, and the time processing of each input was complete. From this information it is possible to compute the average waiting time and the average time required for the system to process inputs, including waiting times. Also, it is possible to record individual waiting times and group these as an estimate of the distribution of waiting times.

Because of the randomness of the arrivals and holding times, it is evident that a single tracing of five inputs through the system is not sufficient to determine, for example, the average time required for five inputs to pass through the system. Many replications of the experiments would be required to estimate this mean time and the distribution of times required for the system to process five inputs. After a few trials, it is possible to estimate the mean and the variance and arrive at a decision about how many more replications are necessary in order to estimate the mean or the whole distribution to the desired accuracy.

The preceding demonstration not only showed how Monte Carlo procedures can be used in system simulation but also indicated the types of computations performed by digital computers used in some system simulations. Because of the way components were represented, it was possible to proceed in tracing inputs by a purely computational procedure. It is possible to draw flow charts that give step-by-step instructions that can be programmed for a large digital computer. Figure 1 shows the master flow chart used in the demonstration. Figure 2 and Figure 3 show the flow charts containing the instructions that were followed when either of the two event types was the next occurring event in the event list or time status record.

Flow charts such as those contained in Figures 2 and 3 can be drawn by people familiar with a particular system and the reason for the system simulation. Computer programmers can translate these flow charts into computer language. Considerable ingenuity is required on the part of the computer programmer to find ways to put into the available storage all of the instructions and component properties necessary for the simulation of a complex system. Considerable work has been done on generation of random numbers in computers and on sampling from distributions. No effort will be made here to discuss these elegant techniques. The techniques chosen for the demonstration could be used in a digital computer simulation but are not elegant.

The purpose of the preceding paragraphs was to give a demonstration of how one might go about setting up and operating a computer simulation of a simple system. The following steps are necessary in such an effort:

1. Become familiar with the system, its components, and the rules of operation.

2. Determine what questions are to be answered by the simulation.

3. Determine what data are to be collected during the simulation. That is, determine what information must be printed out by the computer in order to answer the questions that are to be answered.

4. Decide on the representation of components. For example, in the demonstration we represented the operator and his equipment by a distribution of holding times and a probability of making scrap.

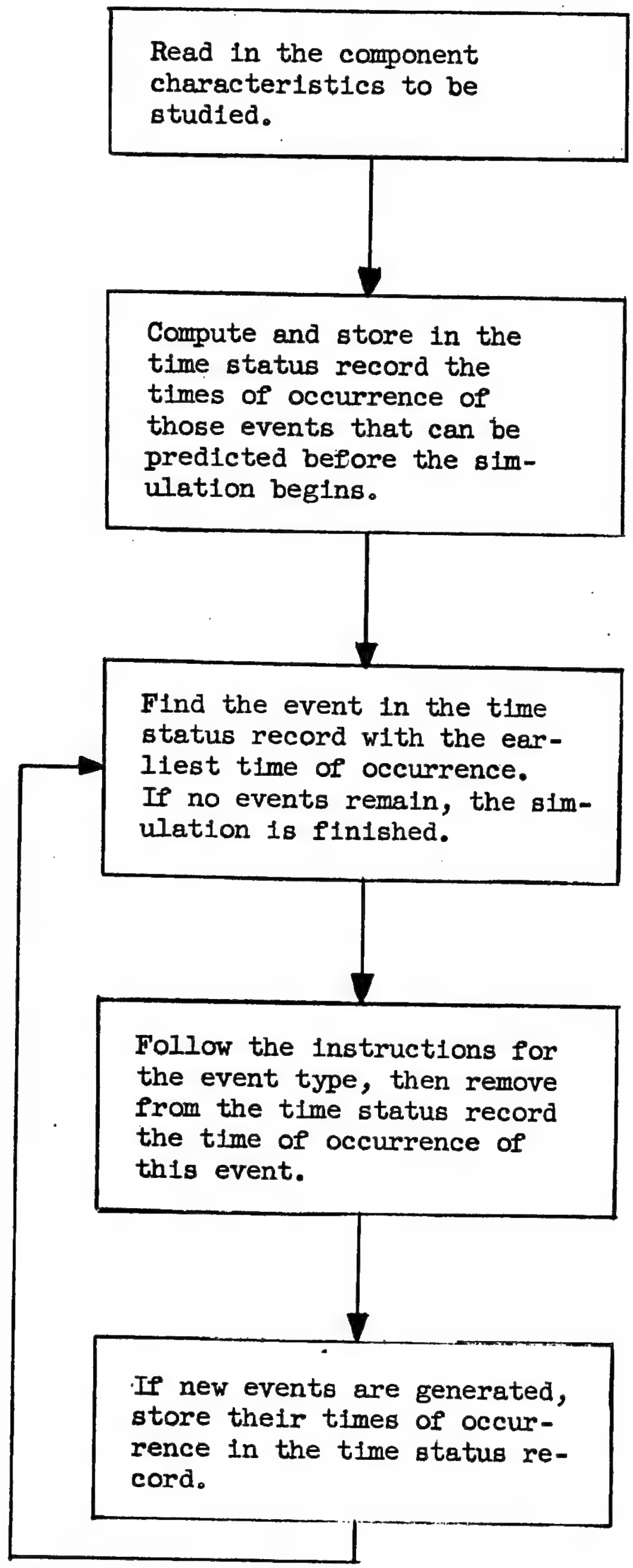


FIGURE 1

Master flow chart for computer simulation in which system is characterized by what happens when only certain important events occur.

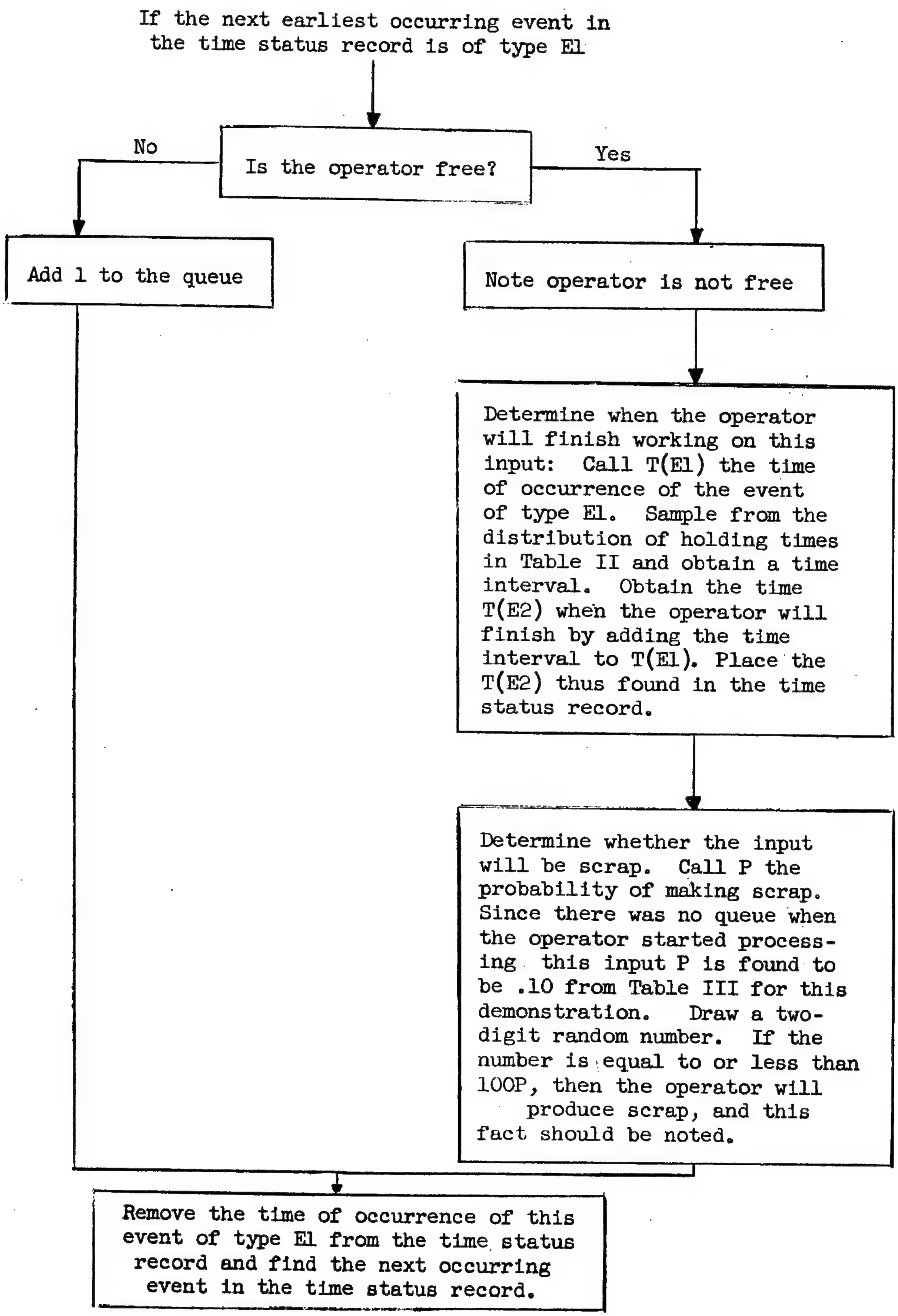

If the next earliest occurring event in the time status record is of type E1
Is the operator free?
No
Yes
Add 1 to the queue
Note operator is not free
Determine when the operator will finish working on this input: Call T(E1) the time of occurrence of the event of type E1. Sample from the distribution of holding times in Table II and obtain a time interval. Obtain the time T(E2) when the operator will finish by adding the time interval to T(E1). Place the T(E2) thus found in the time status record.
Determine whether the input will be scrap. Call P the probability of making scrap. Since there was no queue when the operator started processing this input P is found to be .10 from Table III for this demonstration. Draw a two-digit random number. If the number is equal to or less than 100P, then the operator will produce scrap, and this fact should be noted.
Remove the time of occurrence of this event of type E1 from the time status record and find the next occurring event in the time status record.

FIGURE 2

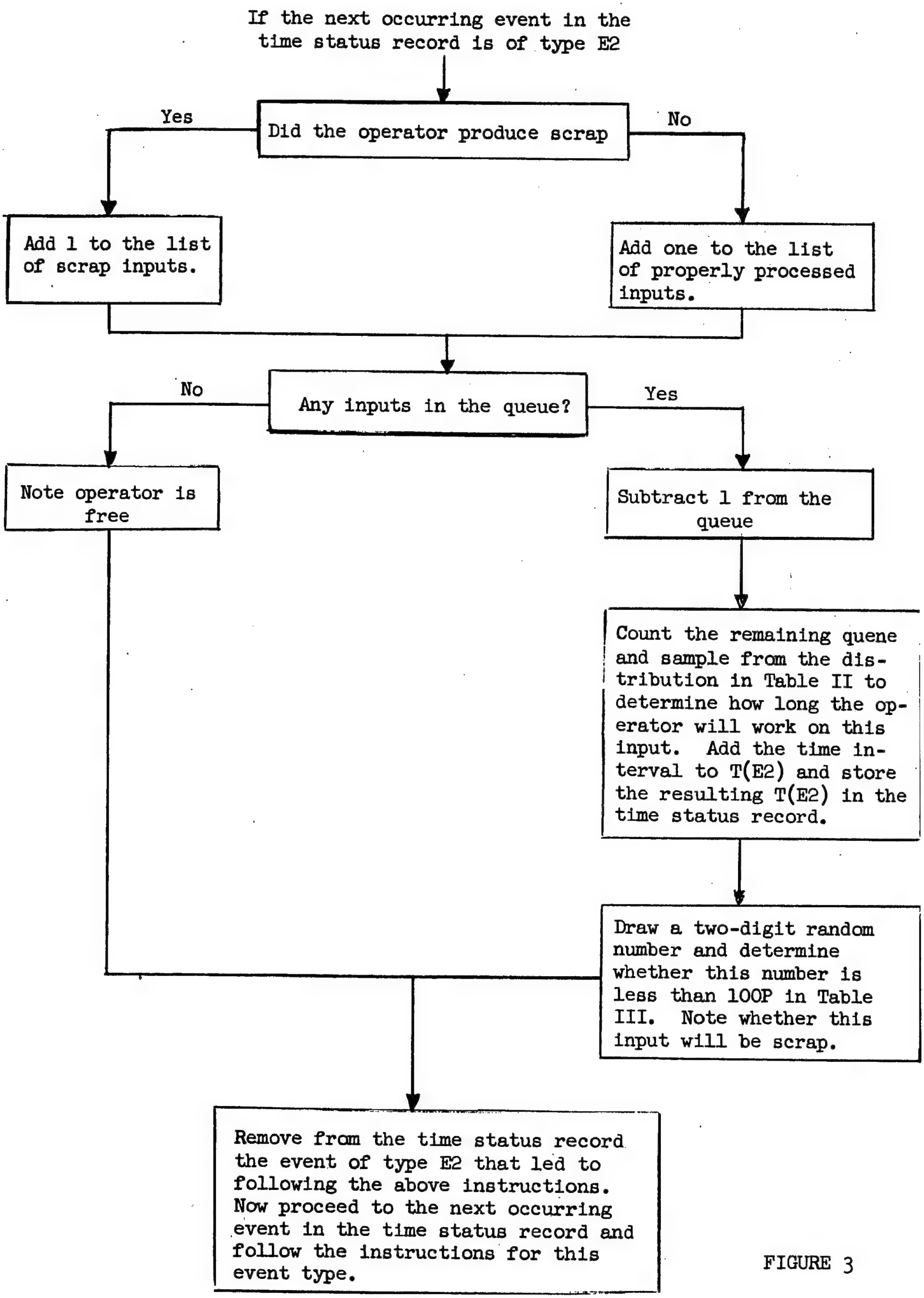

If the next occurring event in the time status record is of type E2
Did the operator produce scrap
Yes
No
Add 1 to the list of scrap inputs.
Add one to the list of properly processed inputs.
Any inputs in the queue?
No
Yes
Note operator is free
Subtract 1 from the queue
Count the remaining quene and sample from the distribution in Table II to determine how long the operator will work on this input. Add the time interval to T(E2) and store the resulting T(E2) in the time status record.
Draw a two-digit random number and determine whether this number is less than 100P in Table III. Note whether this input will be scrap.
Remove from the time status record the event of type E2 that led to following the above instructions. Now proceed to the next occurring event in the time status record and follow the instructions for this event type.

FIGURE 3

5. Draw the flow charts, showing how inputs are to be handled and what is to happen when particular events occur. For those systems or those components where the investigator cannot write the rules to cover all interesting possibilities, then a computer model is not an appropriate model.

6. Program a suitable computer to follow the instructions contained in the flow charts.

7. Obtain the component characteristics that must go into the model. For example, distributions of arrival or holding times must be obtained either by measurement or by theoretical studies. Frequently more effort goes into determining component characteristics than in any other part of a system simulation program.

8. Design experiments to be run on the model of the system. While this was not demonstrated, many costly computer runs may be avoided by proper use of statistics in determining the most efficient way to answer the questions to be answered.

9. Replicate experiments on the computer a number of times as required to obtain results of required accuracy. Sometimes as few as ten replications will permit an estimate of the mean that meets the accuracy requirements, but probably several hundred replications will be necessary to estimate distributions.

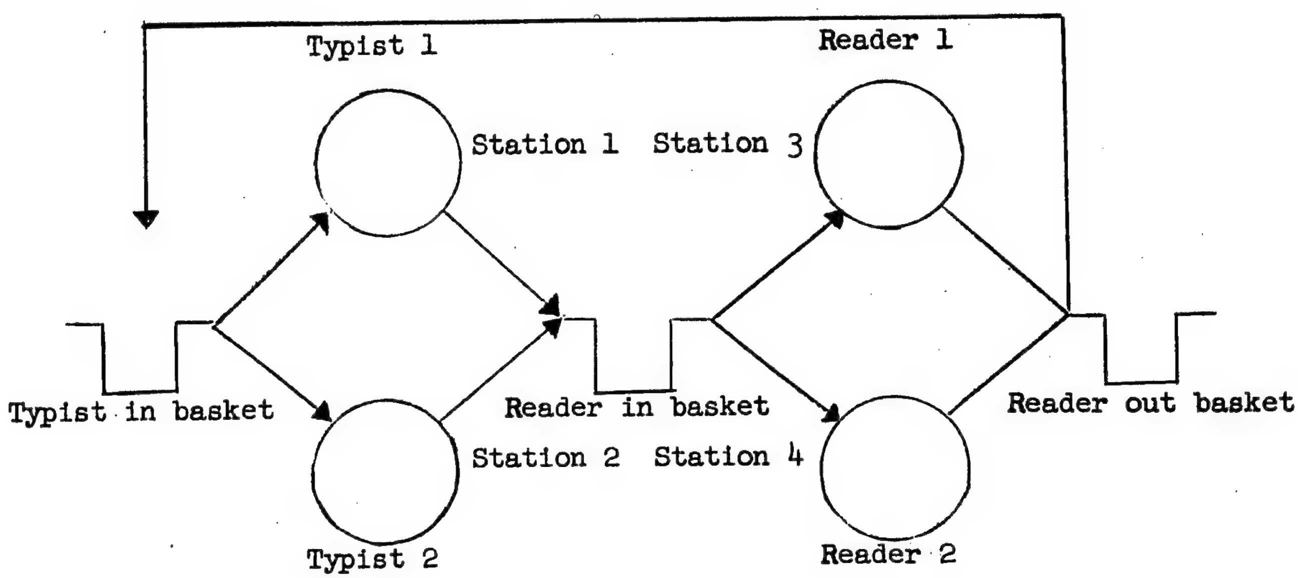
10. Devise experiments to test the ability of the model to predict system performance.

Not all of the above steps can be demonstrated here, but all are important. Space does not permit discussion of each step despite the importance of each.

The purpose of the rest of this discussion will be to show how flow charts can be drawn for more complex systems, taking into account more factors such as equipment reliability, feedback, and series and parallel operation. While the system to be considered next is only slightly more complex than that already shown, no attempt will be made to show how instructions are followed and how inputs are traced through the system because of the large number of pages that would be required. Also, no system of really interesting complexity will be considered because merely presenting the flow charts for an interesting job shop scheduling problem or for an interesting air defense system would also require too many pages. However, the flow charts for the slightly more complex system will give a better feeling for the complexity of these flow charts and the extent to which the same set of flow charts can be applied to different systems. It will be clear, for example, that changing component characteristics is possible in going from one set of runs to another and that the flow charts, therefore, have considerable generality. That is, in some cases the flow charts for an air defense system may be useful without change for studying missile systems and manned interceptor systems.

It will also be evident, however, that changing rules of operation require changing the flow charts; and this is the extent to which the flow charts lack generality. If the alternative rules of operation of interest are known beforehand, it is possible to program the computer so rules can be readily changed.

THE SIMPLE SYSTEM CHOSEN FOR ILLUSTRATION OF FLOW CHARTING.



The figure above shows a simple system that can be used to illustrate in what sense a system can be characterized by what happens at the occurrence of certain important events. The figure is also useful for discussing the component characteristics and rules of operation of the system that must be known if the system is to be studied by the use of a computer simulation.

The figure might be the block diagram for a system consisting of two typists and two proofreaders. The typists receive rough drafts of one page reports containing considerable tabular data. The proofreaders check the numberical data in the reports for errors in typing.

When new work arrives for the typists, it is either handed to a free typist or is placed in the in-basket if both typists are busy. Both typists take work from the same in-basket.

When a typist finishes typing a report, she either hands the report to one of the proofreaders if one is free or places it in the in-basket shared by the proofreaders.

There are two types of reports. Type 1 has priority over type 2, but occasionally a typist or a proofreader will select a type 2 report from the in-basket even though both types of report are present in the in-basket.

Occasionally a proofreader finds an error and returns a report to the typists for retyping. No priority is given to reports that require retyping.

Occasionally a typewriter fails or a proofreader's lamp fails. There is one maintenance man. If equipment fails while the maintenance man is busy, it must wait for service. The maintenance man selects at random one of the requests for service when he is free.

When a typewriter fails, the typist removes the report from her typewriter and places the rough draft in the in-basket. That is, the report will have to be completely retyped by herself or the other typist. When a proofreader's lamp fails, the proofreader places the report on which he was working in the in-basket. That is, the proofreading will have to be done over by either one of the proofreaders.

Work placed in the in-baskets following equipment failure is not given priority over other work.

The paragraphs just above show the amount of description necessary for even a simple system. Obviously other rules of operations could have been chosen, but the rules described above permit the drawing of flow charts of only modest complexity.

Although we have described a system involving typists and proofreaders, the flow charts will be applicable for answering some questions about a system consisting of two radar operators and two weapon assigners. Stations 1 and 2 could represent two operators who are charged with identification and tracking of aircraft. If new blips appear on the scopes while both operators are busy, they must wait for service. Stations 3 and 4 could represent two officers charged with threat evaluation and weapon assignment. When both officers are busy, newly identified targets must wait for service. Occasionally weapon assigners find obvious mistakes in identification and return targets to the radar detectors for re-entry into the system. If equipment fails while any one of the operators is working on an input, the input must return to the waiting line and be completely reserviced by another operator. Priority of targets could be based on position. Not included in the flow charts to follow, however, are the instructions for carrying out the computation of positions of aircraft.

Typical questions that might lead to modeling the system are:

1. How will daily production of reports be affected by better training of typists? Presumably the training will result in a different distribution of service times for the typists and a lower probability of making an error.

2. How long will it take to get reports out if we double the number of reports to be typed each day? Presumably, increasing the rate of arrival or otherwise changing the distribution of arrivals will affect the time spent by reports in waiting for service.

3. If we buy electric typewriters, what will be the cost of producing a finished report? Presumably new typewriters are being considered because they will shorten the typist service times and lower the probability of error. The cost of a finished report can be determined from the usual costs plus the cost of new equipment suitably amortized.

These and other questions that might be asked require a means for finding the relationship between the following:

1. Distributions of arrival.

2. Percentage of priority reports.

3. Probability of operator making a selection error.

4. Probability of an operator spoiling an input.

5. Distributions of operator holding times. Note that 3, 4, and 5 will be affected by training and equipment provided the operator.

6. Rules for selecting inputs from among those in waiting lines.

7. Distributions of breakdowns of equipment.

8. Distributions of repair times.

9. Distributions of waiting times and times required to get inputs through the system.

10. Costs of wages, floor space, material, power, and equipment.

A digital computer simulation will permit finding the relationships between the parameters listed above.

We will now proceed to the flow charts for the system. They will not contain instructions for printout but will emphasize the tracing of inputs.

NOTATION AND ABBREVIATIONS IN THE FLOW CHARTS.

To assist in the understanding of the flow charts that follow, the abbreviations and notation used in the flow charts will be explained.

jE1	Is the event of the arrival of an input of type j. If j equals 1, the input is of higher priority than if j equals 2.
ijE2	Is the event that either station 1 or station 2 has just completed processing an input of type j. Thus, i refers to either station 1 or station 2.
kjE3	Is the event that either station 3 or station 4 has just completed processing an input of type j. Thus, k refers to either station 3 or to station 4.
sE4	Is the event of a breakdown of station s where s can refer to any one of the four stations.
sE5	Is the event that station s has just been repaired by the maintenance man.
T(ijE2)	Is the time of occurrence of the event ijE2, and the time of occurrence of one of the other events is designated in a similar way.
WLI	Is the waiting line for inputs to stations 1 and 2.
WL2	Is the waiting line for inputs to stations 3 and 4.
WLM	Is the waiting line for repair jobs waiting for the maintenance man.

While most of the instructions in the flow charts are self explanatory, some may require explanation. For example,

Generate a T(ijE2)

stands for the following set of instructions: Call the time of occurrence of the event of type jE1 present time. Sample from the distribution of service times for this station and obtain a time interval. Add this time interval to present time to find the time when this station will finish processing the input.

The question

Error?

stands for the following instruction: Let p be the probability the station will make an error in selection of an input. Draw a random number from a population rectangularly distributed between zero and one. If the random number is less than p, say the station made an error. If the random number is larger than p, say the station did not make an error in selection.

The instruction

TSR

stands for the following set of instructions: Delete from the time status record the time of occurrence of the event for which the subroutine was just executed. Find the next occurring event in the time status record and execute the subroutine associated with this event type.

Figure 4 contains instructions for handling inputs to the typists. Inputs are immediately serviced if one of the typists is free and go into the waiting line if neither of the typists is free.

Figure 5 shows the instruction for disposing of an input processed by the typists and also contains instructions for simulating reaching for new work if jobs are waiting.

Figure 6 shows the instructions for simulating the handling of outputs from the proofreaders and also contains instructions for simulating the reaching for new work on the part of a proofreader.

Figure 7 shows the instructions for handling inputs when one of the stations breaks down because of equipment failure.

Figure 8 gives instructions for simulating repair and shows how to handle inputs when equipment has been repaired.

Note in Figure 7 that a breakdown leads to removing an event of type ijE2 or kjE3 from the time status record. Thus, some of the events in the time status record are purely tentative.

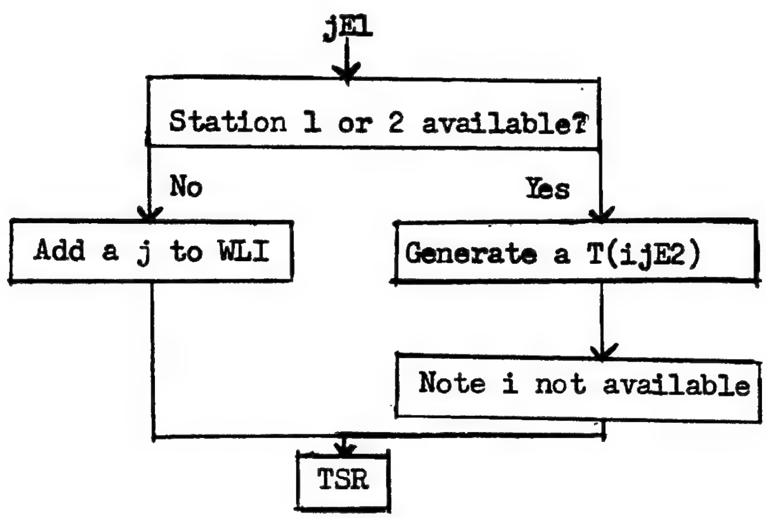
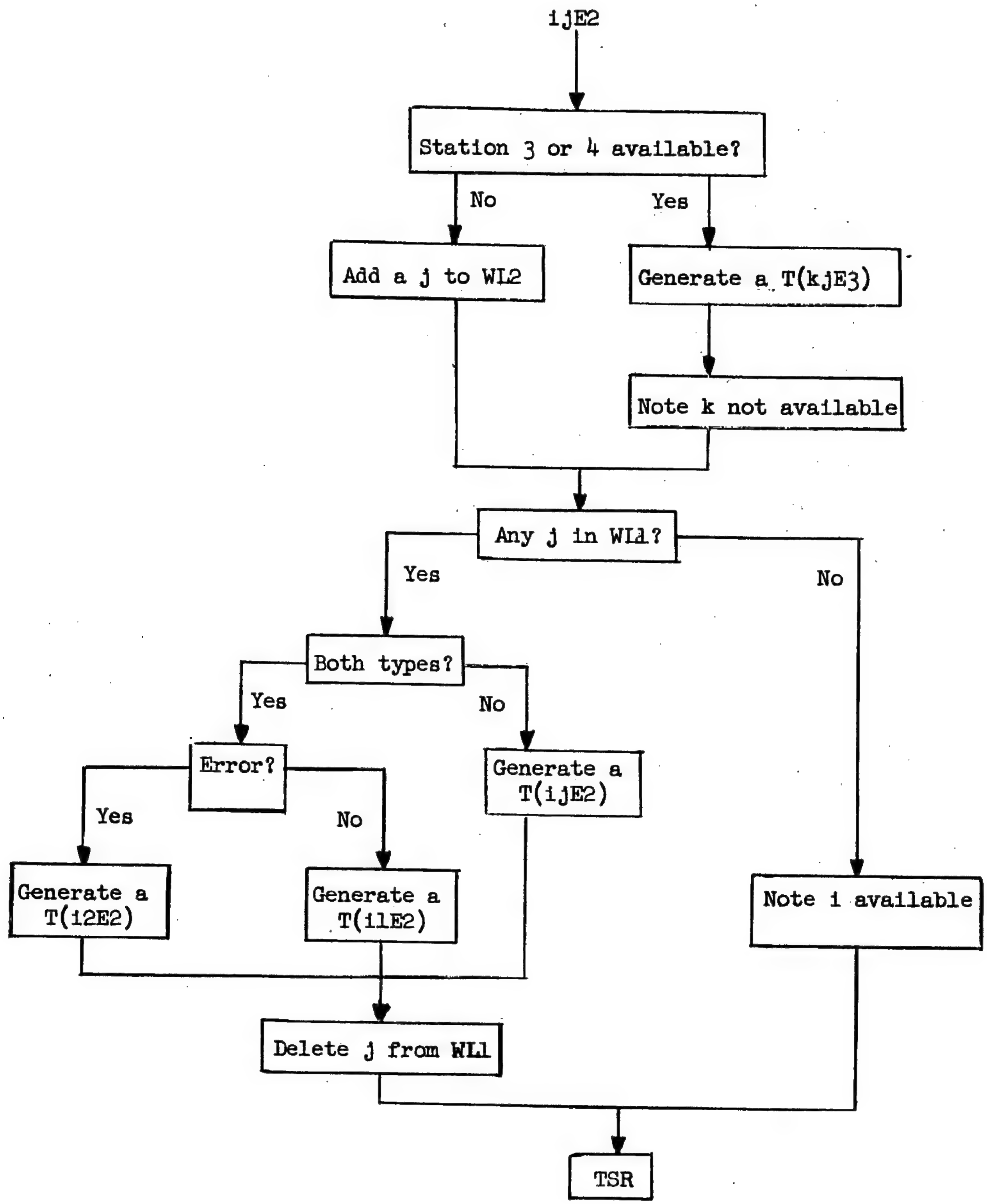


Figure 4



iJE2
Station 3 or 4 available?
No
Yes
Add a j to WL2
Generate a T(kJE3)
Note k not available
Any j in WL1?
Yes
No
Both types?
Yes
No
Error?
Yes
No
Generate a T(ijE2)
Generate a T(i2E2)
Generate a T(i1E2)
Note i available
Delete j from WL1
TSR

FIGURE 5

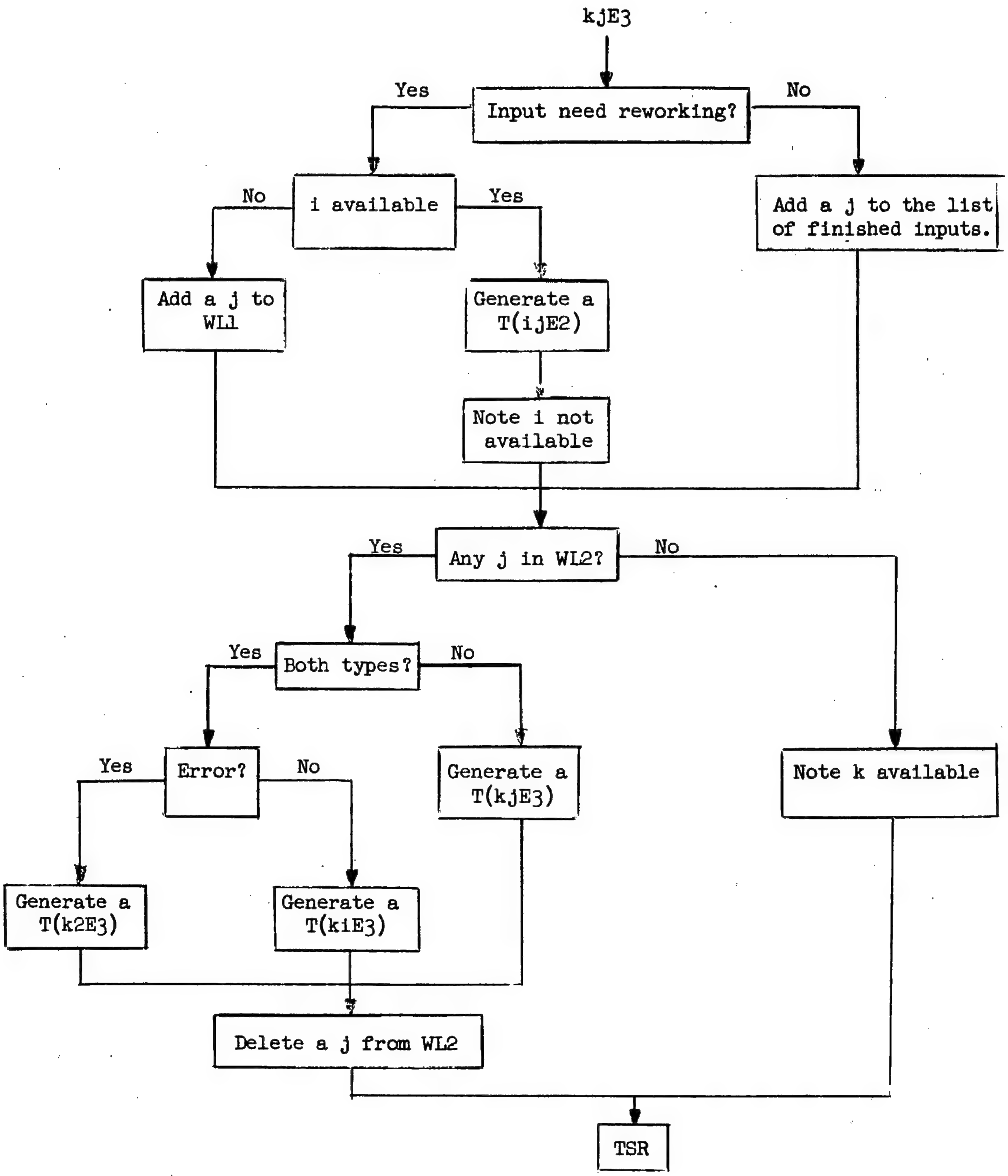

kjE3
Input need reworking?
Yes
No
i available
No
Yes
Add a j to the list of finished inputs.
Add a j to WL1
Generate a T(ijE2)
Note i not available
Any j in WL2?
Yes
No
Both types?
Yes
No
Error?
Yes
No
Generate a T(kjE3)
Note k available
Generate a T(k2E3)
Generate a T(kiE3)
Delete a j from WL2
TSR

FIGURE 6

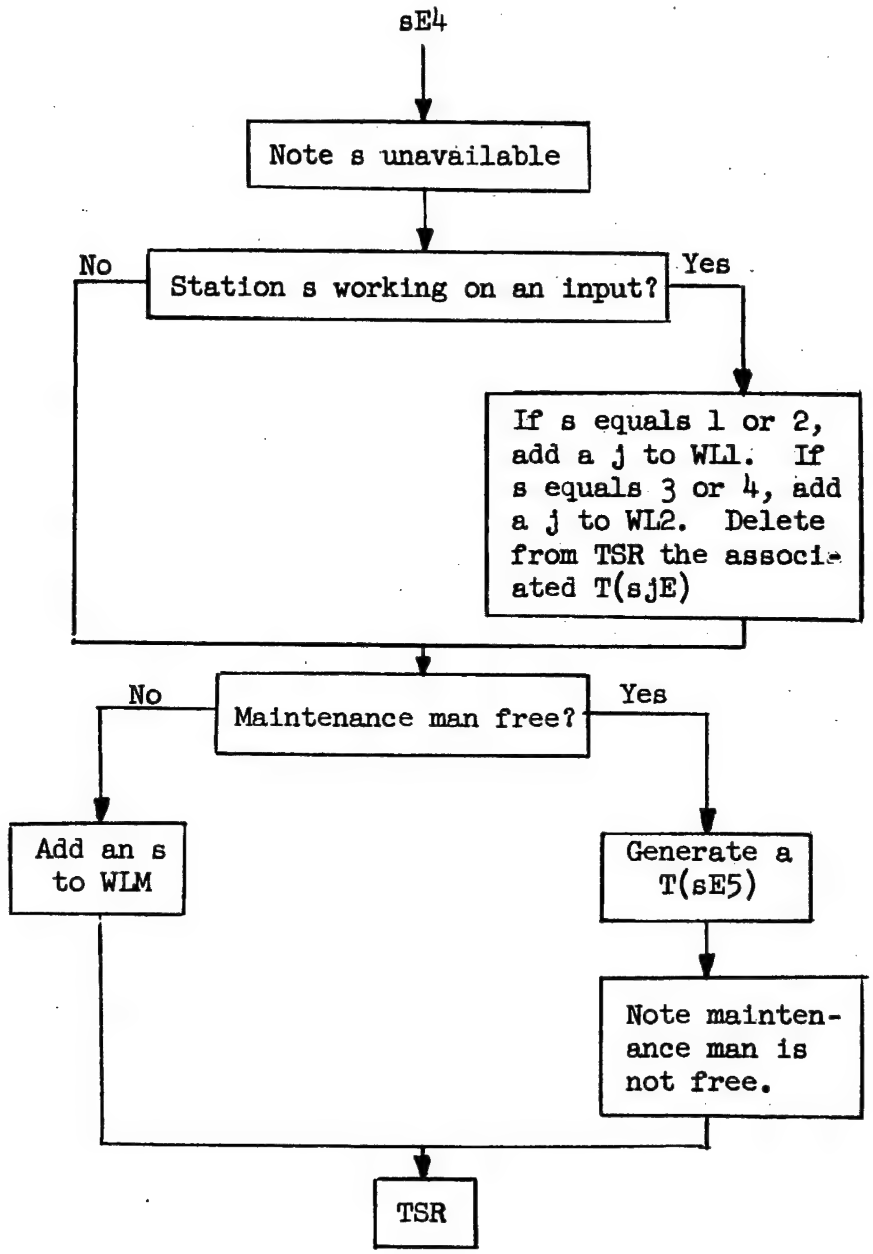

sE4
Note s unavailable
Station s working on an input?
No
Yes
If s equals 1 or 2, add a j to WL1. If s equals 3 or 4, add a j to WL2. Delete from TSR the associ-ated T(sjE)
Maintenance man free?
No
Yes
Add an s to WLM
Generate a T(sE5)
Note mainten-ance man is not free.
TSR

FIGURE 7

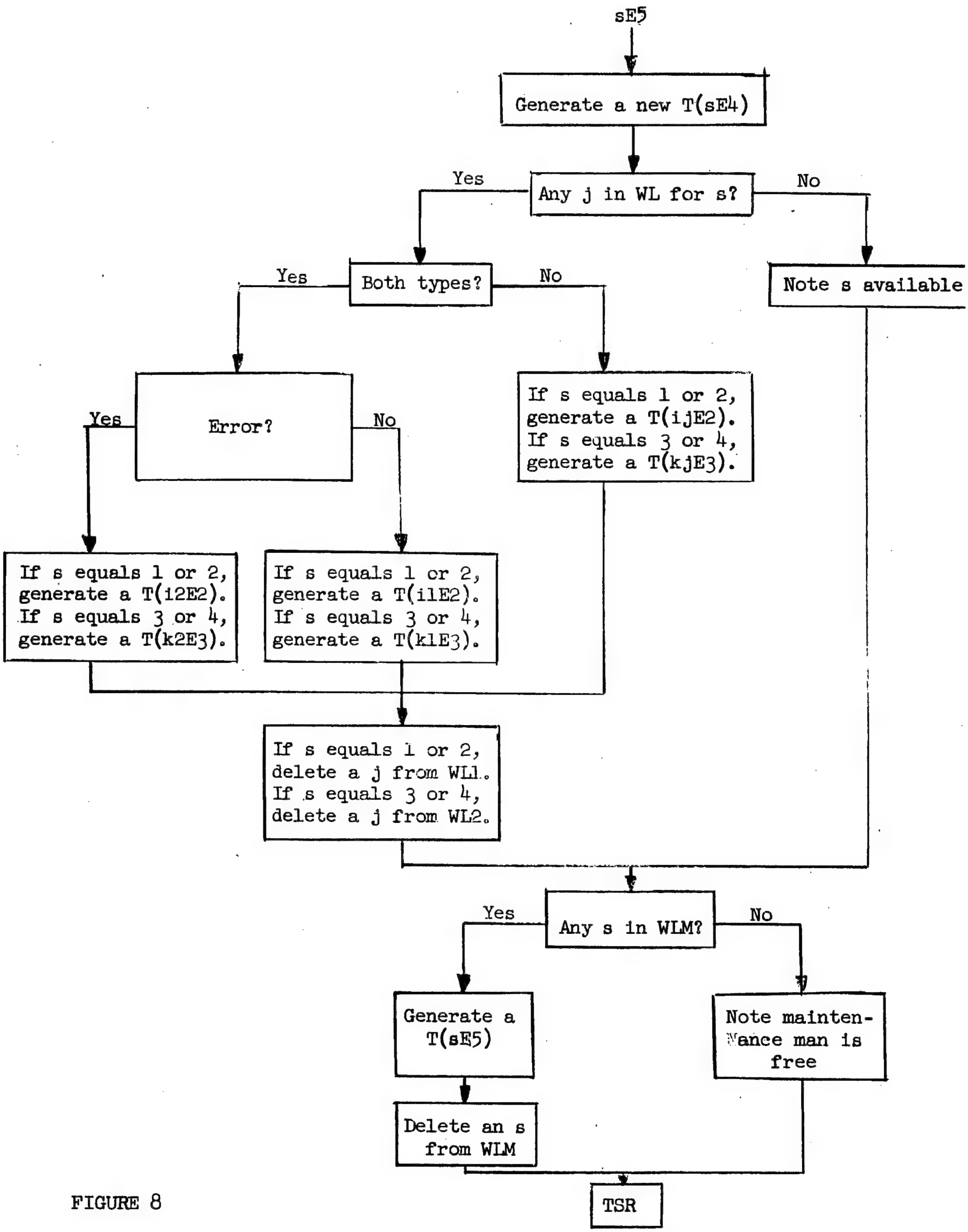

sE5
Generate a new T(sE4)
Any j in WL for s?
Yes
No
Both types?
Yes
No
Note s available
Error?
Yes
No
If s equals 1 or 2, generate a T(ijE2). If s equals 3 or 4, generate a T(kjE3).
If s equals 1 or 2, generate a T(i2E2). If s equals 3 or 4, generate a T(k2E3).
If s equals 1 or 2, generate a T(i1E2). If s equals 3 or 4, generate a T(k1E3).
If s equals 1 or 2, delete a j from WL1. If s equals 3 or 4, delete a j from WL2.
Any s in WLM?
Yes
No
Generate a T(sE5)
Note mainten-ance man is free
Delete an s from WLM
TSR

FIGURE 8

As the flow charts indicate, five event types will be stored from time to time in the time status record. The times of occurrence of events of type E1 can be generated before the simulation begins. Also, whether a particular input is a priority input can be determined in advance from the known probability of occurrence of a priority input. If a large number of inputs are to be studied and there is insufficient room in storage to accommodate all the times of arrival, then it is possible to generate only some of the arrival times before the simulation and generate the rest one by one as times of occurence of these events are deleted from the event list or time status record.

Repaired equipment can be expected to operate for some length of time before breakdown occurs again. The time to break down is a distributed quantity. The first time of occurrence of the event type sE4 for each station can be obtained by sampling from the distribution of time to failure for each station. Thus, before the simulation begins there will be four events of type sE4 in the time status record. Times of occurrence of succeeding breakdowns cannot be determined until equipment has been repaired.

Before the simulation begins it is necessary to know and read into the computer the distributions and probabilities that describe component characteristics.

As stated before, no attempt was made to include in the flow charts instructions for printing out information. Potentially, any of the blocks in the flow charts could call for print outs. In Figure 4, for example, instructions could call for the computer to print out each time an operator becomes free or becomes busy. Such information may be desired in attempting to determine what fraction of time operators are busy. Or the computer might be required to print out the size of the waiting line each time an input is added or removed from the waiting line if information is desired concerning the average length of the waiting line or the distribution of the lengths of the waiting line. So much information can be printed out that the investigator must be careful to resist the temptation to "print out everything." Information should be printed out only when there is a definite use for it. Much computation time can be wasted in printing out needless quantities, and the investigator can soon be engulfed with masses of data that nobody wants to interpret.

In looking back over the examples, I am struck by the fact that I included no examples of the types of technical computations usually required to represent some of the components. These computations are frequently associated with up-dating the positions of inputs or components when position is important. Also, in simulating military systems, such computations are required to determine trajectories of missiles, for example, and impact points or interception points. Thus, the time of occurrence of an event may depend on a technical computation as well as the result of sampling from a distribution.

No special section will be devoted to utilization of models. The remarks leading to the development of flow charts necessarily contained considerable discussion of the uses to which mathematical models are put.

The examples have shown that a wide variety of system components can be represented by numerical quantities and rules for computation to solve many system problems. There still remains many system problems that cannot be solved using the techniques we have discussed mostly because we do not know how to represent components or do not know the rules of operation.

COMPUTER PLAYED WAR GAMES

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The Weapon Systems Laboratory of the BRL uses war games as one of several means of evaluating army weapons and weapon families. The games are designed to be played automatically on the ORDVAC**.

The ORDVAC is a high-speed electronic digital computer. It was built for the BRL several years ago by the University of Illinois. It is a general purpose computer which may be used for solving many varieties of mathematical problems.

In addition to the four elementary operations of addition, subtraction, multiplication, and division, the computer performs other basic operations. The most important of these are (i) duplicating numbers, (ii) moving numbers from one part of the machine to another, and (iii) after testing two quantities for "equality" or for a "greater than" condition, choosing one of two paths to follow depending on the results of the comparison. Every problem to be computed on the machine must in the final analysis be expressed in terms of the basic operations.

The ORDVAC memory is composed of two devices, the magnetic core with a capacity of 4096 words and the magnetic drum with a capacity of 10,032 words. In the memory we can store data and information which, in case of hand computations, are kept in the mind, or recorded in notes, tables, etc. We can store not merely the numerical results of all operations, but also a sequence of orders. Thus, the computer can be automatically sequenced so that it can perform a whole program of computations without any human intervention. The sequence in turn needs a detailed set of instructions--the code--taking into account all the acts of judgment and memory that a person would perform in carrying out the computations by hand.

When needed, the computer can easily generate random numbers distributed uniformly, normally, or according to any simple distributions.

The operating speed of the ORDVAC is quite high. Add time is 50 microseconds (i.e. about 20,000 additions per second) and multiplication and divide times are about 500 microseconds.

Shift time and storage access time of the core memory are about 15 microseconds. The magnetic drum is much slower and is used when there is not enough storage space in the core memory.

Well, so much for the computer used in our war games. Next I would like to take up the procedure used in designing games to be played on the ORDVAC.

Several steps are involved here and the overall time required depends on the complexity of the game. For games designed to date this procedure

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**Ordnance Discrete Variable Automatic Computer

has required about one man year, but for more complex games it might take several man years to complete.

The first step is to define the problem to be investigated and to determine what information--output data--is desired from the game. The Weapon Systems Laboratory, of course, is interested primarily in questions concerning weapons and families of weapons, so that our games are designed with emphasis on weapons' effectiveness.

The second step in designing a game is to prepare a battlefield scenario describing the two opposing forces--their weapons, targets, logistics, and intelligence--and the tactical situation--the time frame, outstanding terrain features, lines of departure and contact and military policy and objectives.

Based on the battlefield scenario, a detailed set of rules is then provided. These rules must be in sufficient detail that play can be carried out for all conceivable situations that might arise during a play of the game. These rules would include all the input data on weapons, targets, logistics, intelligence, movement, terrain, and how these data vary with time and circumstances.

At various points in the play of the game the commanders would be presented with choices from several alternatives and would make their choices taking into account all the information available at that moment. In computer played war games, all the choices that could arise must be provided for before play of the game commences, since it would be impractical to continually start and stop the computer during the play of the game in order to make the decisions as they arise.

Finally, the game must be coded so that it can be played automatically on the ORDVAC. This step takes several months to complete. Because of the complexity of war games most of the core memory's 4096 words are used as storage space. It takes from 3 to 10 minutes to play a game once on the ORDVAC and it is usually necessary to play the game several hundred times to obtain all the desired information. Here, again, the times involved depend on the complexity of the game.

Next, I would like to describe the war games that we have designed to date. So far the War Games Group has devoted its effort to games played at two different levels--the company level and the army level.

The game at the company level was designed for use in comparisons of various infantry rifles. Play begins with the Red assault force moving out from the line of departure. Physically, the line of departure might be a road or stream. The objective of the Red force is a terrain feature, such as a hill or a wooded area. The objective is assumed to be located about 1500 feet from the line of departure.

Red moves forward under the cover of artillery in several waves, one wave moving for a short period of time while the others take cover and give it protective fire. The effect of Red's fire is to inflict casualties on the Blue defender and to suppress Blue's small arms fire. Blue exchanges artillery and small arms fire with Red. The effect of Blue's fire is to inflict casualties on Red and to force Red to advance at a faster rate.

When Red has advanced to about 400 feet from the Blue force, Red's artillery fire is lifted because of friendly safety requirements. When they have advanced to about 120 feet from the Blue force the several Red waves combine into one wave of men and advance at a constant speed of 7-1/2 feet per second. Red now fires from the hip, while Blue continues to employ aimed rifle fire. During this final stage of the assault, hand grenades are also exchanged.

At the start of each game there are 128 Blue riflemen. The size of the Red force is varied from game to game and may be as high as 640 men. In order to facilitate the play of the game on the ORDVAC, the play is carried out in 128 independent engagements, with one Blue soldier opposed by the appropriate number of Red soldiers.

For each engagement, a record is kept of the ranges at which Blue and Red exchange rifle fire and hand grenades, and the ranges at which Red and Blue are killed by rifle fire, artillery fire, and hand grenades. The records of 128 engagements are combined to give the cumulative percentages of Blue and Red casualties against range, and the cumulative number of rifle shots and hand grenades exchanged by Blue and Red against range.

The game ends when one side or the other suffers a sufficiently high percentage of casualties. From cumulative graphs of the Blue and Red casualties the ORDVAC can determine the range at which the game ends, which side wins the game, the number of casualties suffered by each side, and the number of shots and grenades exchanged by Blue and Red.

Several factors are taken into account in the play of the game:

(1) Terrain

(2) Red's speed and maneuver

(3) Size of the targets

(4) Rifle characteristics - lethality, accuracy and rate of fire

(5) Effects of artillery fire

(6) Effects of hand grenades

(7) Number of waves of Red soldiers, and

(8) The ratio of the Red force to the Blue force.

All together 14 groups of 25 games each are played for the same rifle. The ratio of the Red force to the Blue force is the quantity that is varied from one group of games to the next. Of particular interest is the fraction of games won by Blue as a function of this ratio. Curves showing the probability that Blue will win vs. the ratio of Red to Blue can be constructed for each rifle type and used in comparisons of the different rifles.

When this game was being designed it was feared that differences between rifles would be masked by other effects, if the game was made too complex by including many infantry weapons, such as machine guns, automatic rifles, tanks, and so on. For this reason we included only rifles, artillery mortars, and hand grenades, and even then we expected moderate differences to be lost. To date, about 2000 games have been played employing six different rifles having various accuracies, lethalities, and rates of fire, and actually, moderate variations in rifle characteristics have resulted in appreciable differences in the outputs.

This is illustrated in the figure. The center curve was obtained for the M1 cal..30 rifle. The right hand curve was obtained for a rifle 10% more accurate and lethal than the M1, and the left hand curve was obtained for a rifle 10% less accurate and lethal than the M1.

You notice that in the area of interest (in which each side has some chance of winning) the curves for the different rifles are parallel. For this reason the ratio at which each side has an equal chance of winning is a good measure for comparing the rifles' effectiveness. It is seen from the figure that a 10% change in rifle characteristics results in about a 10% change in the rifle's effectiveness.

In games played at the army level, one Blue army opposed a Red force of equal or greater strength. Studies at this level are conducted by the War Games Group for one of three reasons:

(1) To compare two families of weapon systems

(2) To compare two weapon systems

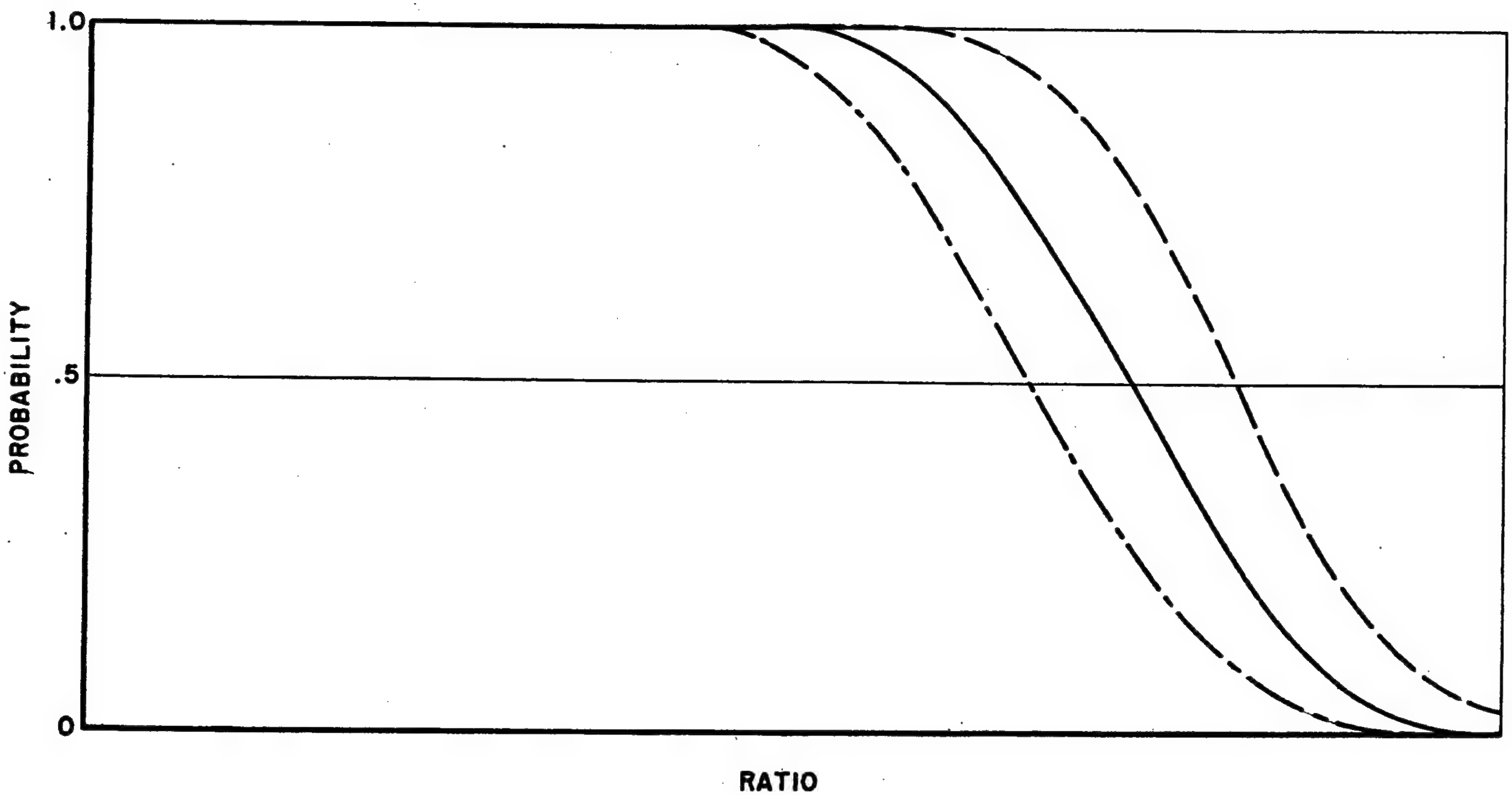
(3) To show a need or lack of need for a certain type of weapon system.

Two basic models are used in these studies. The first--The Ground War Game--was patterned after a model devised by Northrop Aircraft, Inc. under contract to BRL; the second --ANSWEG* was recently completed by the War Games Group. One of the major differences in the two models is that ANSWEG uses Monte Carlo methods while the Ground War Game uses average or expected values for all numbers.

In these models, a time period such as one hour or one day is established and the computer executes the various steps in this time period over and over. (In ANSWEG the main steps are: Intelligence, Non-atomic Battle, Atomic Battle, and Movement.) One of the difficult problems is to decide how much detail to put into each step. More detail perhaps makes for a better model, but it also can begin to tax the capacity of the machine as well as the human operator who is burdened with the task of gathering a large amount of input data. Moreover more detail in one step can demand more time and more input data in another step. A good example of this can be found in the Atomic Battle step of ANSWEG. Both the Red

*Atomic and Non-atomic Support Weapons Evaluation Game

BLUE'S PROBABILITY OF WINNING



and Blue forces possess a number of targets that are attacked by the enemy with atomic weapons. Similar targets are grouped together into a single target type. Now it seems logical to assume that, up to a point, more target types result in a better model. However, more target types also increase the machine time of the game and the amount of input data since, for every target type, the following questions must be answered:

(1) How easy is it for the enemy to find one of these targets?

(2) What enemy weapon systems will fire at these targets?

(3) How do these targets move on a battlefield?

(4) When one of these targets is killed, how is the performance of that side affected?

The last question has been the most difficult to answer even for the most obvious atomic targets. It is, of course, an easy matter to state the actual function of, say, a communications center. But when a center for one side is "knocked out" there must be a provision in the model to somehow penalize that side. If there is no such provision, the enemy will be better off in the game not to fire at these targets.

As was mentioned before, one of the purposes of these war games is to compare two families of weapon systems. This is done by running a game (or series of games with identical inputs) with the characteristics of one family in the model, and then running another game with the characteristics of the other family. Nothing else but these characteristics is changed from the first game to the second. However, for a given study, more than one pair of games is usually compared. Factors such as level of intelligence and available supply of atomic warheads are varied from pair to pair.

Unfortunately, it is not usually possible to run a set of games and obtain as a result two numbers which, when compared, will reveal the better family. One family will undoubtedly show up better by one criterion and the other family will be better by another criterion. Some of the criteria that have been used are:

(1) Cost of launchers, missiles, and warheads expended

(2) Amount of fissionable material expended

(3) Weight of missiles fired

(4) Numbers of casualties

(5) Number of miles advanced

(6) Number of days the war lasts.

If war games were the only method used, it would indeed be difficult to make the final decision as to which family is more desirable. However, war gaming is only one of several methods used by the BRL to arrive at a final decision.

In planning for new generations of war games, a number of problems still remain unsolved by the War Games Group. The basic problem is this: How can war games be made both simple and realistic? Specific problems are:

(1) How can logistics, intelligence, mobility, and terrain be introduced into computer played war games in a realistic way? and

(2) How can strategy and tactics be introduced realistically into the games and still keep the number of battlefield decisions at a minimum?

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ASPECTS OF OPERATIONS RESEARCH IN MISSILE SYSTEM TESTING

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ABSTRACT. The required pace of Weapon System Development, the unit cost of flight tests, and the engineering complexity of future missile systems, present a problem in maintaining a sufficiently complete and current comprehension of the technical nature of a system among those who require such knowledge. Testing is viewed as the primary source of information about a developing system. Mathematical models are proposed as the routine vehicle for the current understanding of the system, and are represented as the heart of the testing operation. Present-day "deterministic" flight simulation models are briefly described. Future probablistic models are discussed. The over-all problem of advancing test capability is suggested for more thorough OR treatment.

1. INTRODUCTION. In recent years we have witnessed a rapid growth of the so-called "system concept" or "system viewpoint" in various fields of engineering. The national challenge for efficient development of effective weapons has lent special impetus to the evolution of this concept. Of course, the nation's weapon and space vehicle projects have been most notably influenced by its application.

Basic to the system concept is the emphasis on interactions and interdependencies of the component subsystems. The engineering objective in general is to create or adapt components which, when made to perform together, produce a timely, economical, and reliable fulfillment of the intended system purpose.

Maintaining a working comprehension of this purpose and of the component interactions, during the development of a large, complex system, presents a formidable problem to the developing agency. Component design activities require a comprehensive knowledge about the system as well as detailed knowledge about adjacent components. Program managers and decision makers require concurrent information concerning system capability, cost, and availability. There are many activities which require current information about the growing system in terms of logistics, safety, training, compatibility with other systems, etc. This general problem of maintaining a working comprehension of the system and its purpose among participating agencies according to need, invites extensive attention and deliberate application of tenets not unlike those associated with the field of operations research.

While Operations Research has done much for missile system planning, the author believes that it can also speed up the missile system development process by contributing to improved test operations. Extensive use of mathematical models as "vehicles" for the current and working understanding of each large system, and incorporation of advance in-flight measurement techniques constitute the major part of the needed improvement.

The evaluation of test operation, of course, provides the quantitative data for each system. Admittedly, much of the missile development process is what has been called in OR literature "ill-structured," i.e., not amenable to known OR computational techniques. However, good OR work is characterized by consideration first of the problem rather than the technique. A potential OR field is introduced here together with some proposed concepts and a brief discussion of some of the associated technical research currently underway.

The present work has arisen incidental to the Ordnance Mission of Missile Systems Testing at the White Sands Missile Range, New Mexico.

2. TESTING PROVIDES THE FEEDBACK. Intelligent system management and program decisions require measurements of the developing system that continuously estimate the capability, suitability, and reliability of the system. Test results of the designed hardware serve as sound bases for the needed estimates for both subsystem and overall system tests.

In general, testing involves evaluation and system analysis. Evaluation determines the quantitative information characteristic of the realized system and its parts. System analysis determines the functions and interactions of the system parts. Improved methods of expressing and maintaining the current understanding of complex systems is therefore an essential part and product of an advanced test operations.

The missile system development process can be briefly outlined as follows: (1) an agency initiates a problem statement based on a need and support of an authority, (2) research discovers and creates methods and devices to realize the desired system and subsystem capabilities, (3) testing reveals existing inadequacies within subsystems and between subsystems and further, suggests solutions for removing the inadequacies, and (4) after all modifications are completed the system is operable thus providing the solution to the problem initiated by the agency.

The problem statement mentioned above usually consists of a compilation of the desired capabilities and characteristics, together with a functional block diagram illustrating what the system is to do, and the basic interconnections between the subsystems. Progress in subsystem design has considerable influence on the problem statement.

Perhaps the most vital output of the test program is the feedback information essential to the efficient progress of the design and development stage. The feedback information must lend a stability and direction to the process. Thus, in the system development process, testing is the feedback.

To aid visualization, Figure 1 breaks down the system into blocks with division into system tests and subsystem tests. Component and laboratory tests will be considered as subsystem tests. Figure 1 also shows closed loop connections of these tests with the mathematical model and its readout device, the simulation computer.

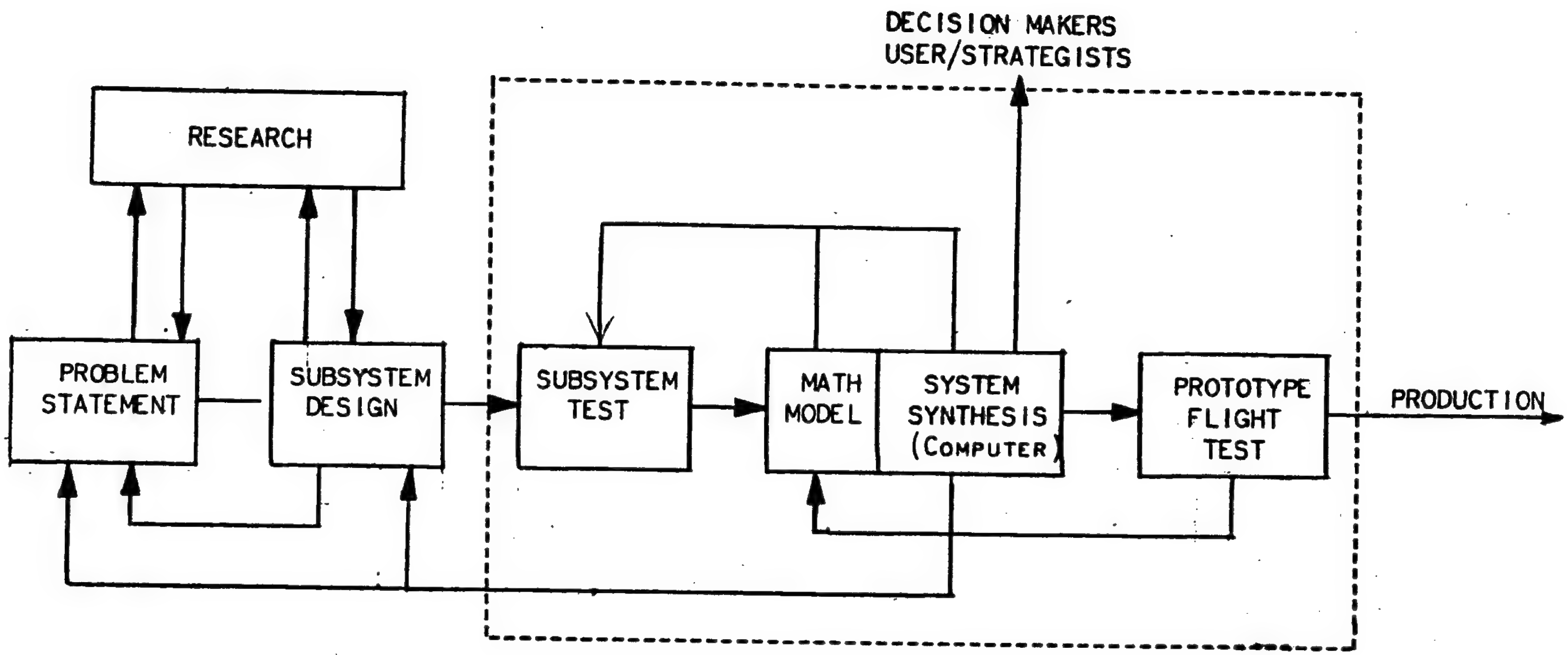


FIGURE 1

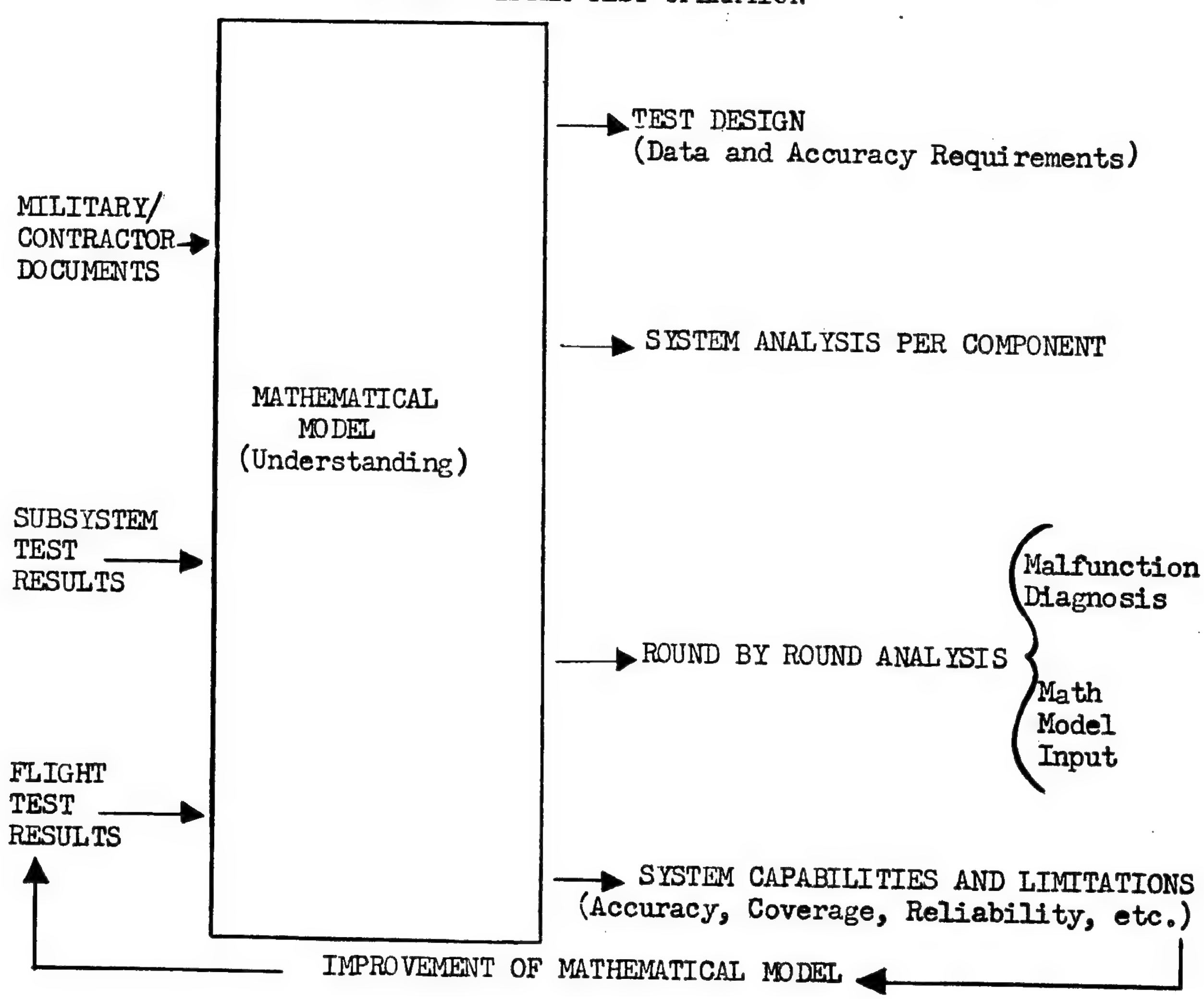
TESTING IS A PRIMARY FEEDBACK MECHANISM IN THE SYSTEM DEVELOPMENT PROCESS.

3. MISSILE SYSTEM MATHEMATICAL MODELS. To conduct the missile system test operation an "understanding" and definition of the tested system is necessary. Generally, in the past this understanding has been largely in the mind of the test engineer. It may be expressed in different forms including equations, diagrams, graphs, etc., but it must be more and more specified and definitive in order to accommodate and promote advanced and accelerating methods of missile and space vehicle development. It must be expressed as much as possible in quantitative or mathematical terms. The model of the system provides a productive and useful end for the numbers which are evaluated from the test measurements. It also integrates the otherwise piece-meal information about the system.

The model is the heart of the test operation. It begins existence as the functional block diagram which is constructed in the problem statement stage of the development process. It can be used in the subsystem design stage for obtaining optimal design parameters and for determining allowable component tolerances. Fig. 2 indicates the inputs to, and uses of the model. Every test experience with the developing system should be directed to updating and improving the missile system mathematical model. The current model,

Figure 2

A GROWING EXPLICIT UNDERSTANDING OF THE MISSILE SYSTEM IS CENTRAL TO THE SYSTEM TEST OPERATION



in turn, is instrumental in designing the tests and interpreting their results. Based on the understanding of the system, studies may be completed which set forth the needed tests, the data required, and the accuracies required. Although this sounds like a bootstrap procedure, it is actually a process of learning, in which new facts are based on the old. In this case, the memory which is kept up-to-date according to hardware test experience is comprehensive and less volatile. It is defined with respect to assumptions, tentative information, and approximations. It is accesible in detail for study, and also is readily adaptable to computer mechanization for solution of system performance and of over-all capabilities and limitations.

Improvements in the format for the mathematical model and in the procedures for its use, of course, are the subject for continual research. Experience at the White Sands Missile Range has involved several types of diagrams including verbal block diagrams in which descriptive words are used to label data flow and subsystem function, and mathematical block diagrams in which transfer functions are displayed. A possible development will be the "correspondence table" in which each system variable and parameter is listed together with complete, appropriate information concerning its location in the actual field hardware, its mode (Electrical modulation, voltage range, etc.), its noise characteristics, etc. Opposite this information in the table would be corresponding information with respect to the mathematical simulation of the variable (including precision, approximations,etc.) A third column of information may be included in the table concerning the transducer and measurement characteristics of the variable.

It is emphasized that the purpose of the model is served primarily during the development, i.e., while the system is growing and changing. A final complete mathematical model is no longer useful for its primary intended purpose.

Associated with needed research toward improved formats and procedures in connection with the mathematical model,is that for models which efficiently convey the test output information to the various recipients. Considerable attention must be given to defining measures, for example, of the system merit which are appropriate to higher command, measures of component sufficiency appropriate to designers, and measures of maintainability, operability, and safety, for the user.

As procedures become more clearly defined, perhaps along the lines suggested by Figure 2, the entire operation submits itself more and more to automation. It is tempting to extrapolate to the future situation wherein immediately at the completion of each flight test, all the information inherent in the completed test is contained in the up-dated mathematical model.

4. MISSILE SYSTEM MATHEMATICAL MODELS. Mathematical models of missile systems, as applied in engineering testing, have been somewhat different from the models common in operations research. The purpose of simulation, i.e., of exercising models, is the same, however. The purpose is to generate "runs," or simultaneous histories of the variables, which are consistent with known or assumed inputs and consistent with characteristics and interactions of the component subsystems. The "runs" may be intended to show over-all system characteristics as they result from the input and/or component characteristics, or they may be intended to reveal the effects upon the component

subsystem as they result from the desired over-all system performance. Computer programming of the model enables convenient experimentation and study of the system.

Mathematical flight simulation has grown primarily from the mechanization of integro-differential equations on analog computers. The design of servo-mechanisms and automatic control systems has for many years involved simulation techniques, both for analysis of observed performance and for synthesis of proposed characteristics. Dynamic analogies have been employed wherein electrical, or other convenient physical quantities, are made to respond according to the same mathematical equations as those which govern the response of the simulated system. Thus, in missile flight simulation, the dynamics, the motions consistent with the physical forces and reactions, are generated through mathematical analogies. The components of the analog computer can be considered as mathematical operators (integrators, multipliers, etc.). Similarly, a given component of a missile system may be considered to operate upon its inputs to provide its respective outputs.

The response of a fin servo, an autopilot, a tracking radar, or an accelerometer, for example, are simulated through operational analogies described by transfer functions. In addition, extensive missile kinematics are included in the present-day flight simulations. That is, the continuous computation of quantities such as position, angles, velocities, accelerations, etc., as they appear in different, relatively moving reference frames, are included to enhance the utility of the simulation model. The type of mathematical model being described, and presently in wide use, is called "deterministic" or "rigid." Its solutions are repeatable and it does not, in general, contain stochastic variables. More realistic and more useful models are being developed in which the stochastic element is inserted. The ultimate model will provide statistical system experience, through simulated flights, which will closely predict actual missile system in-flight reliability and intercept dispersions.

Steps from the purely deterministic model toward the so-called "probabalistic" model are envisioned and some are already achieved. The deterministic model is used to find the sensitivity to parameter deviations in what are called perturbation studies. A more advanced achievement is represented by the successful insertion of characteristic noise in the deterministic model at one point to find out how it affects other points in the system or to study the manner in which the noise propagates throughout the system. Such noise, for example radar tracking noise, "unclean" booster separation or thrust termination, airframe flexure, etc., has been reproduced or approximated from component or system test data. Figure 3 illustrates how closely this type of simulation resembles the observed results obtained from telemetered records of field flights. Real flight f in motions in this case closely resemble those simulated when typical radar noise is inserted, open-loop, into the simulated system. The next steps to be taken would appear to include closed-loop generation of the environmental variables, then to include simulated susceptibilities of components to the environment. Figure 4 displays the probabalistic model which generates missile performance, generates coherent in-flight environment, and which feeds back statistical component susceptibility in terms of failure or of degraded performance. A final addition might be an "initial condition" block in which

REAL SYSTEM vs. SIMULATED SYSTEM

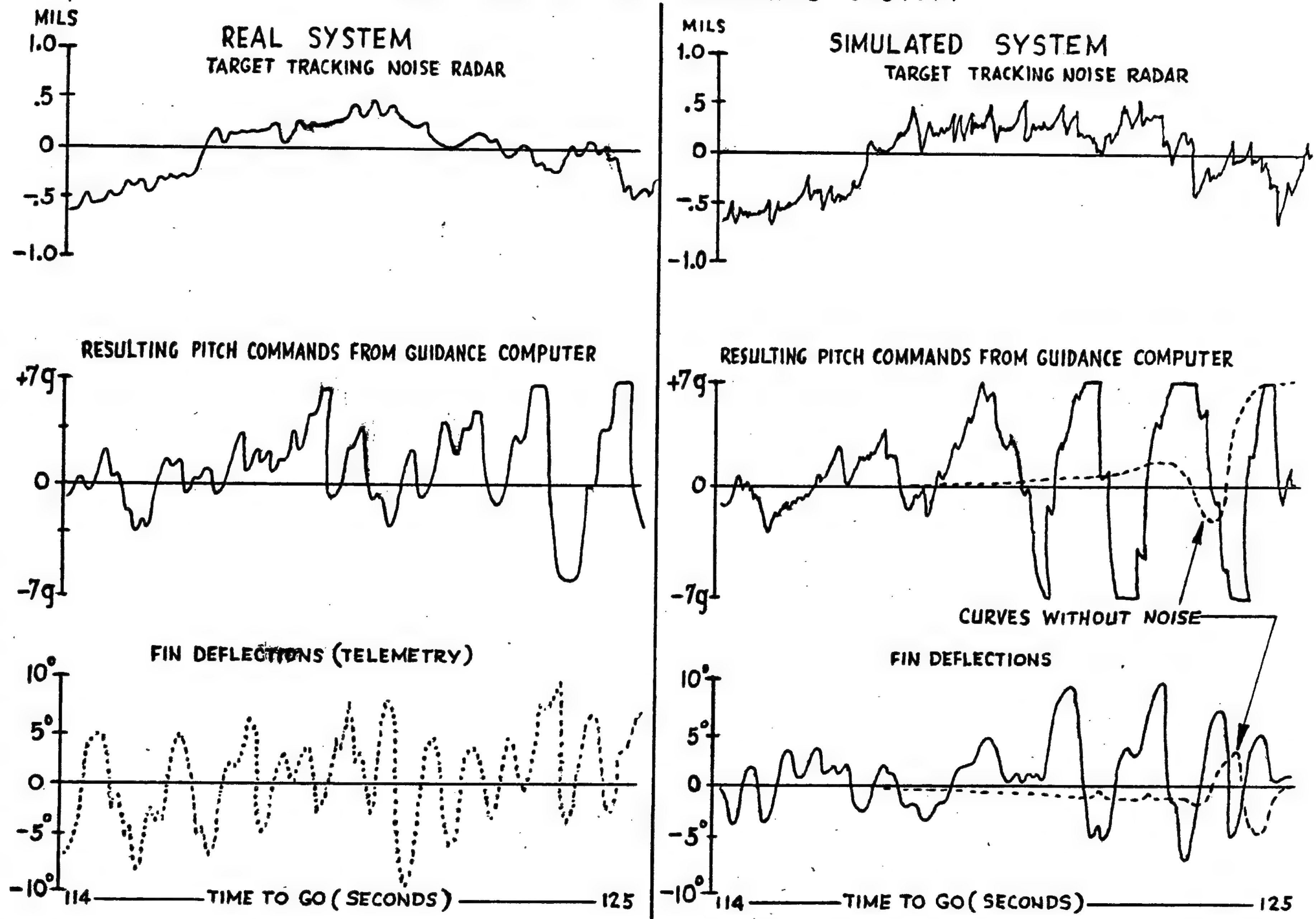


FIG. 3

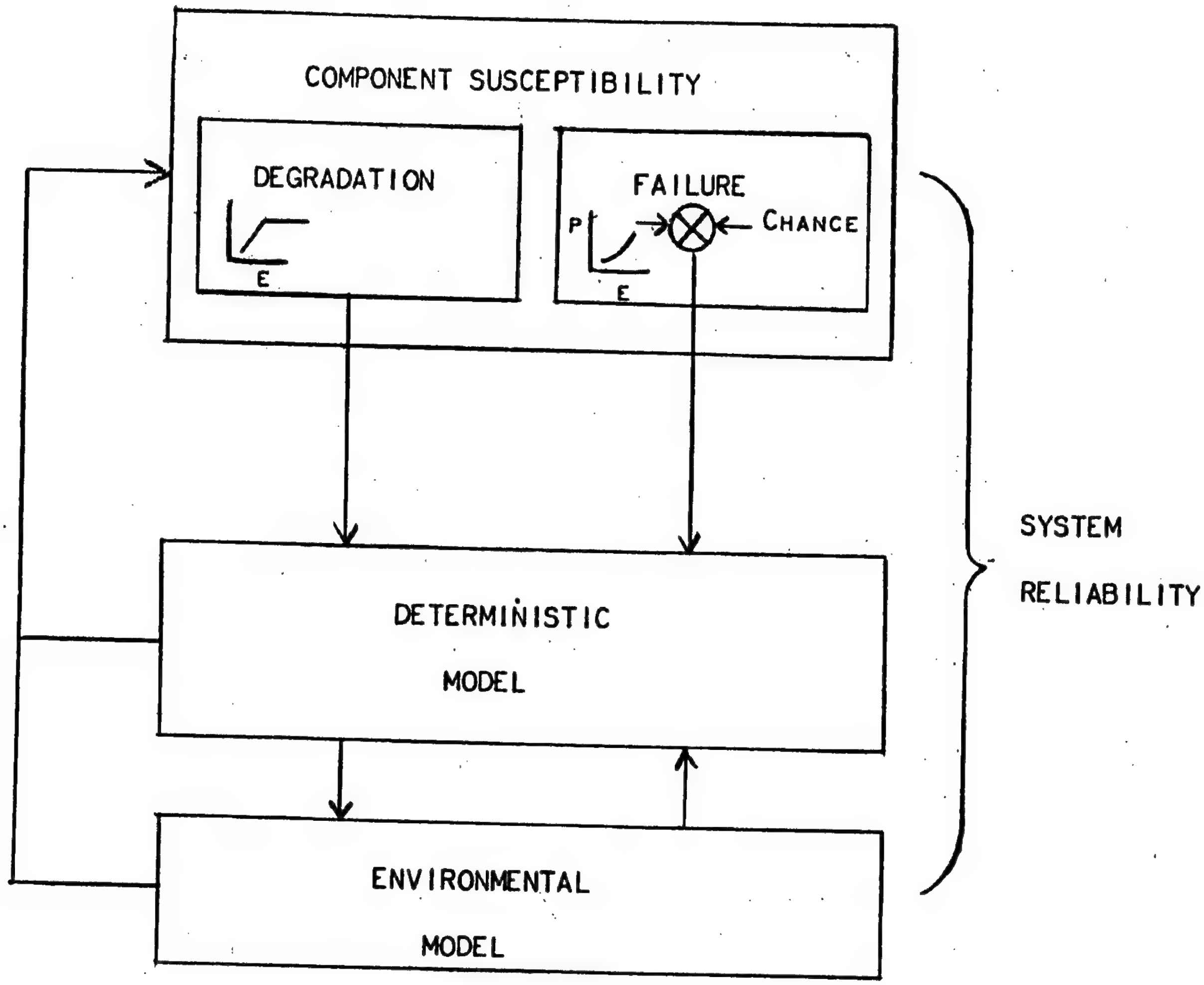


FIGURE 4.
AN ILLUSTRATION OF A SYSTEM RELIABILITY MODEL

calibrations done by humans before the flight, component susceptibilities as they depend on preflight handling, storage, etc., are selected from appropriate probability distributions.

Since simulation serves as a powerful tool in synthesizing or integrating knowledge of component subsystem characteristics toward knowledge of over-all system characteristics, it is felt that research in quest of the probabalistic model will contribute significantly to the existing problem of improving in-flight reliability of missiles and space craft. Simulation developments should complement concurrent developments in environmental testing equipments and in advanced flight test techniques. Again, the model can be the basic tool with respect to designing and interpreting flight and laboratory tests.

5. REAL-TIME FLIGHT ANALYSIS AND TEST CONTROL. The modeling approach to missile system testing provides for a positive communication and identification of system characteristics. It contributes to efficient direction of complex system developments. And, as has been mentioned, it prepares the way for increased automation. Each test flight of a large missile system inherently contains a considerable amount of information about the system. Time and cost considerations force us to glean as much of the inherent information as possible from each flight test experiment. The time involved in gleaning the information obtainable from a missile flight (or hardware test of any kind) and determining its meaning with respect to system objectives is an influential part of the technical lead-time associated with the weapon development. Automation quite certainly will reduce this time more and more. In addition to this, however, there is a "step advantage," which will accrue when automation arrives at the point allowing flight analysis and test control to be conducted in real-time, namely during the flight. It is the advantage of influencing the course of the experiment according to the results of the analysis and evaluation, and according to the immediate in-flight circumstances.

Reference is made to Figure 5. The privilege of selecting monitored missile variables to be telemetered according to need would increase the efficient utilization of flight test data links. Also, real-time data reduction such as missile-borne frequency analysis would increase the amount of data obtainable per unit of data link capacity.

The automatic improvement of the mathematical model through real-time closed-loop adjustment of the simulated parameters is visualized with the aid of Figure 6. Consider a flight test in which the primary objective is to automatically adjust certain specified coefficients in the mathematical model according to the performance being manifest in the actual flight. The idea is basically simple in that the flight analysis and test control facility compares measured quantities in the flying system with corresponding quantities generated by the missile system simulator. Detected differences are employed as error signals in adjusting servo mechanisms which cause the simulated coefficients to assume the values corresponding to their effective counterparts in the flying system. The analysis and control facility employs control on the flying system to acheive appropriate test conditions. For example, an experiment planned at WSMR, in which given aerodynamic coefficients are to be evaluated, will employ commands to the missile which serve to increase the resolution of the computations. It may be desirable to force conditions such that selected mathematical terms are made equal to zero while a given measurement is being made.

Fig. 5

REAL-TIME FLIGHT DATA PROCESSING

- EARLIER TEST RESULTS
- INCREASE INFORMATION CAPACITY OF DATA LINKS
- CLOSED-LOOP MODEL IMPROVEMENT
- TEST CONTROL, OF

MISSILE SYSTEM	TEST SYSTEM
Dynamic Test Sequencing	In-flight Instrumentation Control
Malfunction Diagnosis	Drone Vectoring
In-flight Design Adjustments	Instrumentation for Acquisition and Tracking
Alternate Usage of "Failed" Flights	

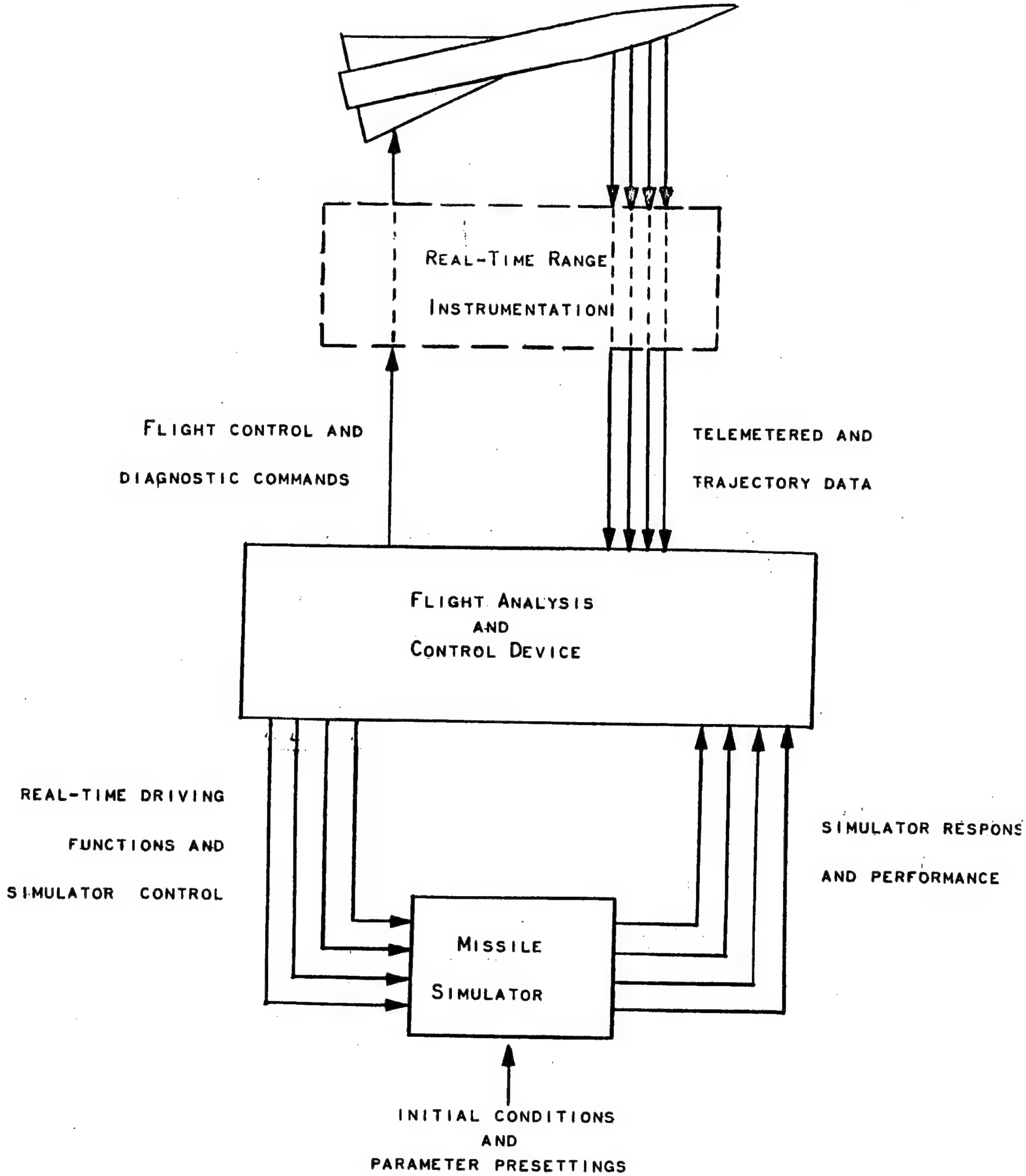


FIG. 6. SIMPLIFIED DIAGRAM FOR REAL-TIME FLIGHT ANALYSIS.

The automatic improvement of the simulated system is a great step nearer an ultimate capability wherein every experience with the system hardware clearly extends the understanding of the system and results directly in conclusions concerning the next step in the weapon development cycle.

The last item listed in Figure 5 refers to the application of real-time flight analysis for the purpose of test control. Note that the control of the test may be directed at the missile system which is under test or it may be directed at the range system which is performing the test and measurement. Illustrative of the latter case are the assigning and assisting of data acquisition instruments and the vectoring of drone craft as targets. The items associated with controlling the tested system are of primary interest and are those which excite the imagination most strongly from the point of view of gaining more system data from the test flight through advanced system test techniques. The opportunity to influence the sequence of test phases as the flight proceeds, increases the potential output of normal flights. The privilege of initiating a series of diagnostic commands which immediately localize a malfunction and furnish additional data before the missile is lost is also important. Future advances may allow certain remedial measures and/or redirection of the flight test to achieve alternative objectives. Many of these items take on a greater significance when considered in connection with very expensive future testing of space craft.

6. CONCLUDING REMARKS. The discussion has briefly described certain researches and how they fit together in a possible forward-looking prespective of advanced missile system testing. The field requires much further study and action in order to maintain a system testing capacity commensurate with the radically new dimensions of future weapon systems. It is felt that the over-all problem is appropriate for treatment by operations research. Certain resulting conclusion may have a beneficial impact on existing R&D policies, such as perhaps far-reaching measures toward insuring that a proposed weapon system be "testable." However, it is anticipated that the greater impact would be the acceleration of meaningful research and implementation of advanced test techniques.

DISCOUNTED LEAST SQUARES

R. J. Duffin[1] and Th. W. Schmidt[2]

This note describes a method of extrapolation of data by fitting with polynomial functions. The method permits use of polynomials of arbitrary degree. However, for the sake of being definite, let us suppose that the fitting is done with second degree polynomials of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 . \tag{1}$$

It is supposed that the data consists of observations given at a regular sequence of values of x, say x = 1, 2, 3, ... The observations for these values of x are denoted by y_1, y_2, y_3, ... Then the central problem is to extrapolate this sequence of observations to obtain a predicted value at x = 0. This extrapolated or predicted value will be denoted by y_0^*. Thus if the polynomial p(x) is suitably fitted to the data, the extrapolated value is given by $y_0^* = a_0$.

A common method of fitting is to select an integer r, termed the <u>range</u>, and to employ a least squares fit of p(x) by minimizing

$$E = \sum_{n=1}^{r} \left[y_n - p(n) \right]^2 . \tag{2}$$

Instead of this procedure, we minimize the expression

$$E = \sum_{n=1}^{\infty} \theta^n \left[y_n - p(n) \right]^2 . \tag{3}$$

We term this method <u>discounted least squares</u>. The constant θ is termed the <u>discount factor</u>. In a qualitative way the discount factor in (3) corresponds to the range in (2). Like the range, the discount factor must be selected to suit best the character of the data. This selection requires judgment. However, typical values would be r = 20 and θ = 0.8.

The notion of discounted least squares was suggested by certain problems in the analysis of missile trajectory data.[3] For example, from a sequence of observations relating to the position of a missile it might be desired to predict the point of impact. In such applications, x corresponds to negative time, and the sequence y_n is usually termed a time series.

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[2] Chief, Systems Research Office, Office of Ordnance Research, United States Army

[3] "A simple formula for prediciton and automatic scrutation," by R. J. Duffin and Th. W. Schmidt, presented to the American Rocket Society, Flight Testing Conference, Daytona Beach, Florida, March 24, 1959.

The method of discounted least squares is a general method, and we believe it should have various applications besides that of rocket trajectory analysis. In particular it should suit certain problems of economics and operations research for which the progressive discount of the past seems natural.

When the minimization of (3) is carried out, a formula of the following type results:

$$y_0^* = \sum_{n=1}^{\infty} Q_n y_n \quad . \tag{4}$$

Here Q_n denotes coefficients which depend on θ but not on the sequence y_n. We term (3) the <u>long formula</u>. Actually the long formula is infinitely long, but the coefficients Q_n decay exponentially and are sensibly zero for large n. This also indicates that practically the sequence y_n does not have to be infinite. Roughly speaking, the practical length of the long formula is a small multiple of the range when using method (2).

The coefficients Q_n have an explicit algebraic expression. This expression will not be given, because in most cases there is an alternative prediction formula which has many advantages over (4). Thus the <u>short formula</u> is

$$y_0^* = 3(y_1 + \theta\delta_1) - 3(y_2 + \theta^2\delta_2) + (y_3 + \theta^3\delta_3) \quad . \tag{5}$$

Here $\delta_k = y_k^* - y_k$ is termed a <u>discrepancy</u>. By y_k^* we mean the predicted value of y_k based on the previous values $y_{k+1}, y_{k+2}, \ldots$ The numerical coefficients in (5) are simply the binomial coefficients of $(a - b)^3$. More generally, if the prediction had been based on cubic polynomials, then the coefficients in (5) would be obtained from $(a - b)^4$.

Formula (5) is called the "short formula" because the predicted value is given as a function of the last three observed values and the last three discrepancies. The "shortness" is a consequence of using memory. The memory is selective because only the pertinent six numbers are used rather than the whole infinite time series.

An important feature of the short formula not manifestly evident is that prediction can now be carried as far into the future as desired. For example, suppose that y_1 is actually not known. Then in the formula y_1 is simply replaced by the predicted value y_1^*. The general rule, then, is that if the observed values are not known, they are to be replaced by predicted values.

To employ the short formula it is first necessary to select a starting index, say k. Thus y_k^* is to be the first predicted value. The values of the discrepancies δ_{k+1}, δ_{k+2}, and δ_{k+3} appearing in the short formula are not known. To get started let us assume them to be equal to zero. This is equivalent to assuming that the values of y_n for $n > k$ are given by a certain quadratic polynomial. This method of starting has the desirable property of giving perfect prediction if the observations y_n are actually defined by a quadratic polynomial.

There is no difficulty in coding the short formula for a digital computer. Moreover, provisions can be made in this coding for automatic scrutation. By scrutation we mean inspection of data for the purpose of rejection of obvious errors or blunders. In hand calculation, scrutation is often accomplished by looking over a graph of the data. The trend in data processing is toward elimination of human judgment. To this end a method of employing the short formula as an automatic scrutator is as follows. Suppose that by theoretical or empirical means an estimate can be made of a standard deviation of y_n, say $\sigma(y_n)$. Then if

$$|\delta_n| > 3\sigma_n \tag{6}$$

we consider y_n to be a blunder, and y_n is replaced by y_n^*. The factor 3 is merely a conventional value.

A common phenomenon in data processing is a missing or lost observation. In automatic scrutation a lost observation is treated as an "excess error." Thus if y_n is lost it is replaced by y_n^*, and the program proceeds without interruption.

In the coding, provision must be made for the contingency that the discrepancy repeatedly exceeds the prescribed tolerance. To do this an integer w is selected. If at any time relation (6) is satisfied for w consecutive values of n, then all past memory is wiped out and a fresh start is made. This process, coded into the computer, would permit extrapolation of curves made up of parabolic segments. An application would be the tracking of evasive action.

Besides gross errors it is necessary to take into account the effect of small errors. To understand this, consider a sequence y_n such that $y_n = 0$ for all n except that $y_k = 1$. We may regard y_k as a unit error deviating from the true values zero. Substituting this sequence in the long formula gives $y_0^* = Q_k$. The same relation would result, of course, if we used the short formula, because the short formula is merely an identity which the long formula satisfies. Thus it is seen that the coefficients Q_n determine how a single error is propagated into the future. Moreover, the coding of the short formula gives an automatic method of numerically evaluating the coefficients Q_n.

A knowledge of the sequence Q_n is necessary for the analysis of random errors in all the observations as well as the single error discussed above. Thus suppose that all the observations have independent random errors with the same variance. Then the variance in the prediction is reduced by a factor ΣQ_n^2. This sum approaches zero as Θ approaches one. (Increasing the degree of the polynomial tends to increase the sum.) In favorable cases one obtains appreciable smoothing as well as extrapolation.

A PRACTICAL APPLICATION OF THE WAITING LINE THEORY FOR THE OPTIMUM ALLOCATION OF MACHINE REPAIR OPERATORS

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INTRODUCTION. This paper will deal with the practical aspects of applying the Waiting Line Theory to a problem of determining the optimum number of machine repairmen required to repair machines which breakdown in a random manner.

Random demand activities arise in such areas like a) gasoline stations where customers arrive in a random manner, b) toll booth operations or c) people arriving at a restaurant.

The object in allocating people to such type activities becomes one of determining the optimum number of people required to handle a given workload. The aim may be a minimum cost program also taking into account certain elements which are not readily definable quantitatively.

The usual method of establishing reliable labor performance standards on routine repetitive activities has been solved by using direct stop watch study and summarizing the times for each work element. However in the area of random demand activities the method for establishing personnel manning requirements is usually based upon arbitrary judgment. The progress in the practical application of the Waiting Line Theory to random demand activities has been unfortunately limited, probably due to the reluctance of practitioners to get involved in the preparation of mathematical models for their own specific situations. It is important that work in the utilization area be continued and common reason be applied to analyze the net worth of a given situation by developing and/or using waiting line models as is practical and logical.

GENERAL DESCRIPTION OF PROBLEM. The problem under discussion is to determine the optimum number of repairmen required to keep machines in operation to assure that production line downtime cost losses as a result of machine breakdown are a minimum. The area under discussion is the Ammunition Loading area where approximately 40 subassembly lines are continually in operation making complex and difficult manufactured ammunition ranging from illuminating powder packed in a container to the explosive loading of missile warheads. This complex involves at any one time approximately 200 people and 65 machines and is located within an approximate one mile square area. For practical reasons, in view of the nature of the workload and security restrictions, the total area is sub-divided into several homogeneous sub-areas. A sample of two sub-areas is shown in EXHIBIT 1.*

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* Exhibits can be found at the end of this article.

SCHEMATIC ILLUSTRATION OF THE PROBLEM. The problem deals with a multiple channel-infinite universe. The schematic representation of this model is shown in EXHIBIT 2. The schematic shows the arrivals, machines waiting for service with a mean arrival rate α (Exponential Distribution). The mean arrival rate is equal to one over the mean time between machine breakdown. These arrivals (M_n) are being placed in the service channels on first come first served basis. The service channels denoted by (R_n) represent repairmen available to make the repair. The mean service rate is denoted by (γ) which is equal to the reciprocal of the mean service time (σ). Thus $\gamma = 1/\sigma$.

MATHEMATICAL FORMULATION. Several expressions of the multiple Channel - Infinite Universe model used to solve this problem are as follows:

Where M = Number of machines

R = Number of repairmen

σ = Average machine service time

π = Average time between each machine breakdown

$\frac{\sigma}{\pi} = \delta$ = Machine repair & performance index

$P_n(t)$ = Probability of n machines requiring service at a time (t)

$$P_n(t) = P_o(t)\frac{\delta^n}{n!}, \qquad n < R,$$

$$= P_o(t)\frac{\delta^n}{R!\,R^{(n-R)}}, \qquad n \geq R.$$

Since $\sum_{m=o}^{\infty} P_n(t) = 1$

$$E(\text{wait}) = \frac{1}{\pi}\sum_{n=R+1}^{\infty}(n-R)\,P_n(t)$$

ILLUSTRATED COMPUTATION PROCEDURE. A sample illustrated computation procedure, average machine and repairman wait times, for a problem having 5 machines with one repairman is shown in EXHIBIT 3. It is evident that hand computation is entirely too cumbersome, for higher value problems and therefore, for those who have need for such tables, reference is made to the Machine - Serviceman Table prepared by the Scientific and Industrial Computing Laboratory, Kodak Park Works, Kodak Eastman Company or other available sources.

DEFINITION OF PROBLEM. The problem data of sub-area A includes thirty machines with a ratio of three operators per machine. The mean service time is 1 hour while the mean time between machine breakdown is 10 hours. The machine and repair index is 0.1. The wait time values for machines and repairmen using different numbers of repairmen is shown below:

WAIT TIME SCHEDULE
AMMUNITION LOADING PRODUCTION LINE
SUB-AREA A

Nr. of Repairmen Utilized	Repairmen Wait Time	Machine Wait Time
1	.0000	.6333 or 63.33 %
2	.0106	.2745
3	.1536	.0689
4	.3299	.0172
5	.4230	.0106
6	.5460	.0011
8	.6591	.0000

The values in the table indicate that if two repairmen are utilized to service the 30 machines then the average repairman wait time is 1.06% while the machine wait time awaiting repair is 27.45%. Obviously this result should be translated into something tangible. In this instance it is desired to express the percentage figure of each category in a dollar equivalent to determine the minimum cost program. As was stated a machine down for repairs idles an average of three operators. This expressed in dollars means that a 1% average machine wait (based on an eight hour shift) is equivalent to a total loss of $27 per shift. By referring to the table below, the cost program for the each repairmen crew size can be seen.

COST PROGRAM

30 machines

Nr. of Repairmen	Cost of Repair (operator $25/day)	Down Time Loss	Total
1	$25	$1,710	$1,735
2	50	741	791
3	75	186	261
4	100	46	146
5	125	29	154

(cont.)

COST PROGRAM (cont.)

30 machines

Nr. of Repairmen	Cost of Repair (operator $25/day)	Down Time Loss	Total
6	$150	$3	$153
7	175	0	175
8	200	0	200

DISCUSSION. The ideal minimum cost solution to this problem is to utilize 4 repairmen resulting in a cost program of $146/day.

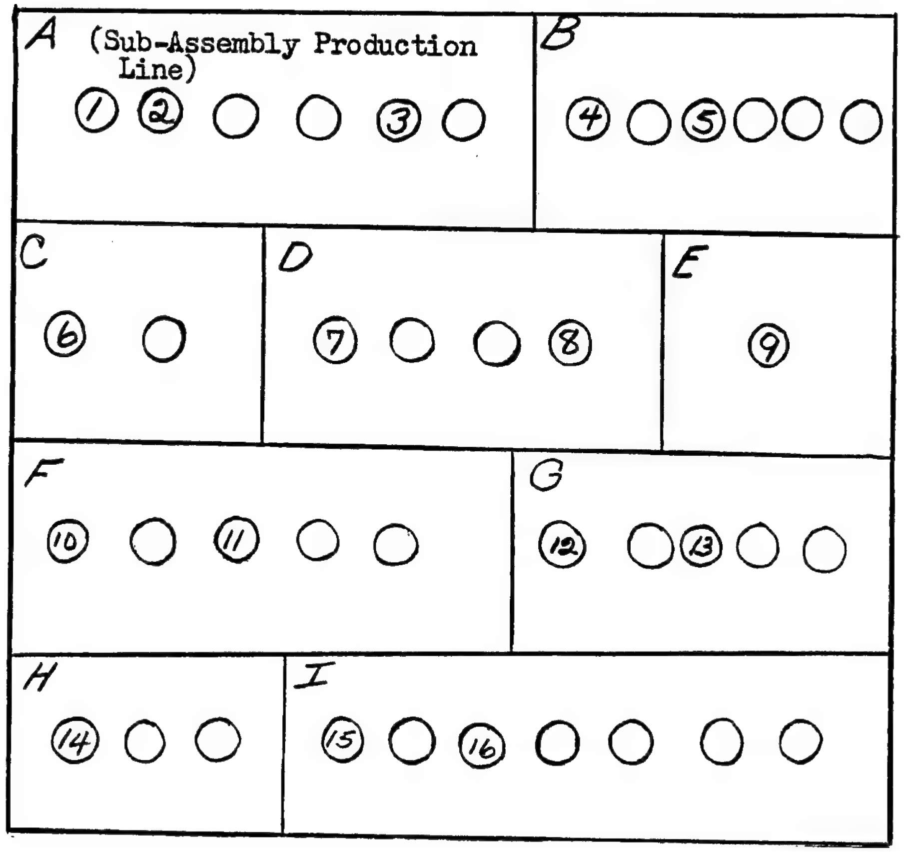
However if the repairmen wait time can be utilized for a preventive maintenance program perhaps it would be advantageous assuming the availability of 2 extra repairmen, to use 6 repairmen. This would add $7 to the cost program but would yield an extra equivalent of 8 hrs. for preventive maintenance (based opon 60% utilization factor of wait time). And of course there would be the extra 1.61% production count.

CONCLUSION. There are many applications of the Waiting Line Theory models to random demand activities within the Ordnance Corps. The results will be fruitful and will provide the basis for continual improvement. Further the coupling of scientific techniques and common reason will yield practical and logical solutions to difficult problems.

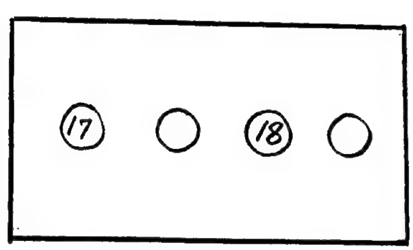
AMMUNITION LOADING AREA

BLOCK DIAGRAM

Homogeneous Sub-Area I



Homogeneous Sub-Area II



Legend

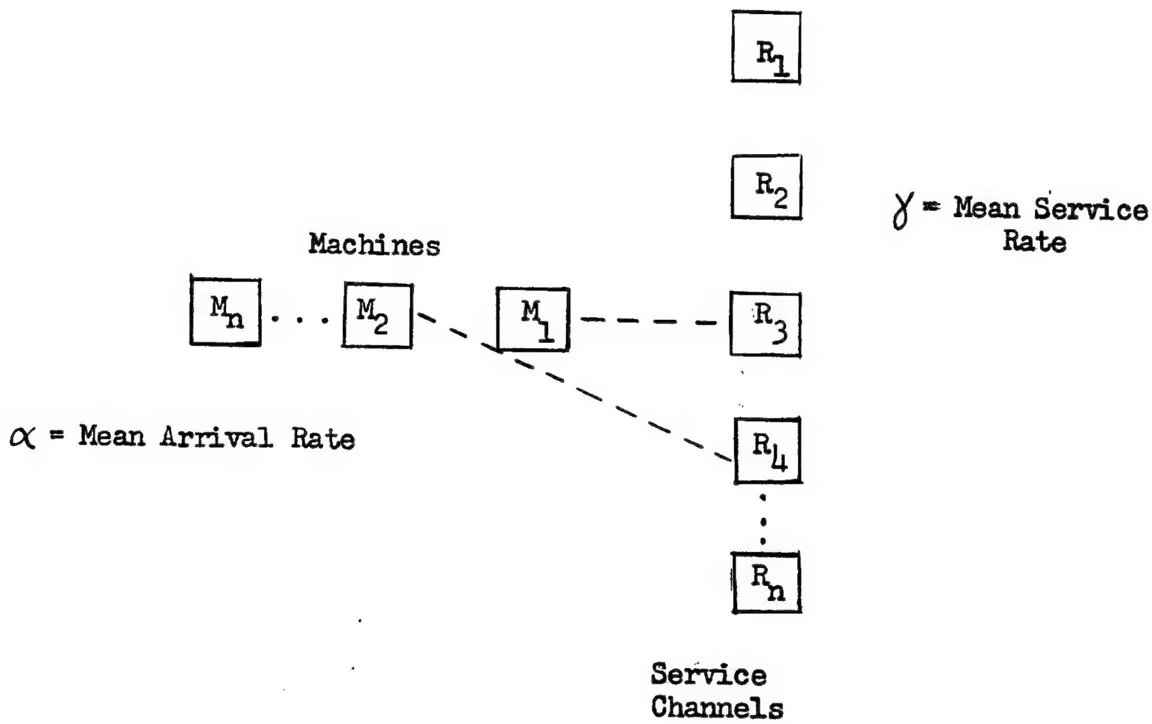
○ = Operator

① - Operator-Machine

A - Sub-Assembly Production Line

Exhibit 1.

SCHEMATIC REPRESENTATION
OF A
MULTIPLE CHANNEL - INFINITE UNIVERSE



$\alpha = \frac{1}{\pi}$ where (π) is the mean time between machine breakdown

$\gamma = \frac{1}{\sigma}$ where (σ) is the mean service time

Exhibit 2.

SAMPLE COMPUTATION PROCEDURE

M = 5 = Nr of Machines
R = 1 = Repairman
δ = 0.1

$\frac{P_n}{P_o} = \frac{M!}{n!\ (M-n)!}\ \delta^n$ where $R \geq n > o$	$\frac{P_n}{P_o} = \frac{M!}{R!\ R^{n-r}(M-n)!}\ \delta^n$ where $R < n$

$$n = o,\quad \frac{P_o}{P_o} = 1.0$$

$$n = 1,\quad \frac{P_1}{P_o} = \frac{5!}{1!\ (5-1)!}(.1) = .5$$

$$n = 2,\quad \frac{P_2}{P_o} = \frac{5!}{1!\ (5-2)!}(.1)^2 = .2$$

$$n = 3,\quad \frac{P_3}{P_o} = \frac{5!}{1!\ (5-3)!}(.1)^3 = .06$$

$$n = 4,\quad \frac{P_4}{P_o} = \frac{5!}{1!\ (5-4)!}(.1)^4 = .012$$

$$n = 5,\quad \frac{P_5}{P_o} = \frac{5!}{1!\ (5-5)!}(.1)^5 = .001$$

= 1.773

$$P_o = \frac{1}{\sum_{m=o}^{m} \frac{P_n}{P_o}} = \frac{1}{1.773} = .565$$

$$P_1 = P_o\frac{P_1}{P_o} = (.565)(.5) = .2825$$

$$P_2 = P_o\frac{P_2}{P_o} = (.565)(.2) = .1130$$

$$P_3 = P_o\frac{P_3}{P_o} = (.565)(.06) = .034$$

$$P_4 = P_o\frac{P_4}{P_o} = (.565)(.012) = .0067$$

$$P_5 = P_o\frac{P_5}{P_o} = (.565)(.001) = .0006$$

Exhibit 3.

Exhibit 3. (cont.)

Probabilities for $M = 5 \quad \delta = 0.1$

n	n-R	R-n	P_n
0	0	1	.5650
1	0	0	.2825
2	1	0	.1130
3	2	0	.0340
4	3	0	.0067
5	4	0	.0006

$$\text{Machine Wait Time} = \sum_{n=R+1}^{M} (n-R)\, P_n \qquad R \leq n$$

$$= \Sigma\, \frac{(1)(.1130)+(2)(.0340)+(3)(.0067)+(4)(.0006)}{5}$$

$$= .04 \text{ or } 4\,\%$$

$$\text{Repairman Wait Time} = \frac{\sum_{n=o}^{R} (R-n)\, P_n}{R} \qquad n < R$$

$$= \frac{(1)(.565)}{1}$$

$$= .565 \text{ or } 56.5\%$$

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LINEAR PROGRAMMING APPLIED TO ECONOMIC SELECTION OF MATERIALS FOR FURNACE CHARGES

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ABSTRACT. Many combinations of furnace input materials exist which will meet a set of given furnace output requirements. The Linear Programming technique can be used to choose that feasible material combination whose material cost is a minimum. The technique can be used to calculate optimum prescriptions for individual heats, or to determine which materials to buy to stock in inventory. This paper is concerned with the "what to buy" decision; a hypothetical problem is solved on a high speed digital computer, and several implications relating to the problem are discussed.

INTRODUCTION. At some point in the operation of a foundry, decisions must be reached which specify the particular furnace input materials (ferro-alloys, bar stock, pig, etc.) to be purchased. The materials which are bought are eventually drawn from inventory, and melted together to form a product of some specified composition and gross weight. It is unfortunately true that the decision concerning "what to buy" is often made without consideration as to the infinite number of possible combinations of different input types (drawn from the catalogues of the many furnace input producers) which will meet the specified requirements. How is one to choose among these many feasible combinations? Does a criterion in fact exist which allows these different combinations to be compared? Obviously yes, since each feasible combination will generally involve a different overall material cost, and this cost should be minimized. It will be shown that the technique called Linear Programming can be used to systematically evaluate the many possible combinations of inputs according to the least-cost criterion, thereby allowing the solution to the best-input-mix problem to be readily obtained.

LINEAR PROGRAMMING. Linear Programming has previously been successfully applied to "best mix" problems as described above. For example, a Columbia economist has computed the least-cost mix of foods that a person could consume and yet meet all the accepted nutritional requirements; farms have applied the method practically in the determination of least-cost cattle-feeds; and the petroleum industry in particular has realized substantial cost savings in applying L. P. (Linear Programming) to gasoline blending problems.[1,2,3]

In general mathematical terms, the technique applies to those processes or activities which are subject to a set of linear restrictions in the form of equalities or inequalities. If the number of feasible solutions within the constraints is large, L. P. can choose that particular solution among all feasible ones which optimizes an objective linear function.

The charging of metals in a furnace is an activity which fits the above description: one could wish to choose the least-cost mix of raw materials which, when melted, will give a product of known total weight and chemical composition. Since each raw material is of different composition, the linear restrictions are those which relate the contents of each chemical element in each raw material to the required content of that element in the end product. If X_i is the unknown weight to be charged of each raw material i, and

P_{ij} is the percent by weight of each element j in each raw material i, then

$$\sum_i X_i P_{ij} \begin{bmatrix} \geq \\ = \\ \leq \end{bmatrix} R P_j$$

where R is the total weight of the heat and P_j is the required percentage of element j. The equations represented by the above symbology are in fact element balances; some appear as equalities, for example the restriction that manganese content equal some specified amount; others appear as inequalities, e.g., the requirement that sulphur (an impurity) content shall be less than or equal to some specific limit; others are also in inequality form, but appear as part of a two equation set stating that the final content shall lie within a specified range.

There is another linear restriction which relates the weight of each raw material charged to the total weight of the heat.

$$\sum_i X_i = R$$

Still other restrictions appear in the metal-charging problem, according to the particular problem being solved. For example, one might limit the utilization of specific raw materials according to their availability. E.g., if home-scrap is used, then the amount generated must limit the amount utilized.

The linear restrictions mentioned above make up a set of simultaneous linear equations and/or inequations* with the X_i's being the unknown variables. Linear Programming extracts an optimum solution to the above set of equations from an infinity of feasible solutions. The condition for more than one unique solution to exist is that the number of unknowns exceed the number of equations. It should be pointed out that whenever an inequality exists, an additional unknown is introduced which converts the inequality to an equality. This unknown is called a "slack variable" since it is introduced to take up the slack (the extent of the inequality) between both sides of the equation.

Once it is established that a great number of possible solutions exist, an "objective" linear function of the unknown variables (raw materials) must be set up to enable the L. P. technique to evaluate the many different possible solutions. Since the objective in this problem is the minimization of cost, a linear expression is contructed:

$$\sum_i X_i c_i = \text{Minimum}$$

---

* An inequation represents an inequality.

where c_i is the cost per pound of the i^{th} raw material.

With the problem formulated, the L. P. computational procedure * is used to extract a solution. The procedure is iterative, requiring a systematic introduction of variables. As much of each chosen variable is brought into solution as the restrictive equations will allow, while amounts of other variables already in solution are adjusted. The method used to choose the variable being introduced at each step causes the results to converge toward a least-cost solution.

THE SOLUTION OF A TYPICAL PROBLEM. A steel manufacturer is producing SAE 310 stainless steel with the following charged composition (furnace losses are added to the required composition, to give the charged composition):

	C	Mn	Si	Cr	Ni	P	S
Percentage:	.25 max	2.00	.75	24-26	19-22	.03 max	.03 max

He carries in inventory the materials shown in Table I, whose compositions and costs are listed. To meet the required end composition, and with the stipulation that generated scrap can contribute no more than 30% of the total charged weight, a melter's calculation gives the prescription per 1000 lb. heat shown in the right hand column of Table I, at a total material cost per 1000 lbs. of $245.55.

The manufacturer wishes to know if he can reduce the average cost per heat by carrying a different set of raw materials in stock. He studies the market, and lists alternate materials to the ones being used. From this point, knowing the compositions and costs of the alternates, the least-expensive combination of inputs that will meet all the requirements can be found by L. P.

* Several algorithms have been developed to solve L. P. problems. The most widely used of these is the simplex technique, and the interested reader may learn the theoretical and computational details of this method from the literature.[4,5,6]

TABLE I

NO. CORRESPONDING TO MATERIALS LISTED IN TABLE II	MATERIALS STOCKED IN INVENTORY	COMPOSITION, % C	Mn	Si	Cr	Ni	COST $/lb.	LBS. REQUIRED IN PRESCRIPTION PER 1000 LBS.
2	FeMn	2.0	82.5	1.5	-	-	.25	22.3
5	FeSi	.15	-	.50	-	-	.20	23.4
9	FeCr	.04	-	1.5	70.0		.41	277.7
11	Ni	.1	-	-	-	99.9	.74	139.1
14	Pig	4.42	.13	.7	-	-	.034	30.1
17	Generated Scrap	.22	1.8	.65	25.0	20.0	.0	300.0
18	Stock	.25	.85	.30	.50	.50	.085	207.4

Total cost of 1000 lb. prescription: $245.55

This was done for a total of 19 materials (which are listed in Table II), including such alloy-rich metals as silico-manganese and ferro-chrome-silicon, two types of bar stock, and several grades each of FeMn, FeSi, Cr, FeCr, Ni. The 11 equations (evaluating 25 unknowns) which are used to solve the problem are listed in Figure 1. The equations are analogous to those symbolically represented earlier with a few additional refinements: each equation includes at least one slack variable with which is associated a high cost (M) in the objective function. These high-cost slacks are inserted for computational purposes only, and will be driven out of solution during computational purposes only, and will be driven out of solution during computation, hence the name "artificial" slack variables. Equations a, c, d, f, g, h, and i contain additional real slack variables, because these equations are in fact inequalities. The cost associated with real slacks is zero, and ergo these slacks do not appear in the objective function.

There are two equations written for the chrome balance, and two for the nickel balance. This is because requirements for these two elements allow a range of possible composition values. For each of these elements, one equation represents the upper limit (content to be less than or equal to x), and the other represents the lower limit (content to be greater than or equal to y).

TABLE II

LIST OF THE 19 MATERIALS WHICH WERE EVALUATED IN THE DETERMINATION OF BEST INPUTS TO STOCK IN INVENTORY.

AVAILABLE MATERIALS	COMPOSITION, %							COST ¢/LB
	C	Mn	P	S	Si	Cr	Ni	
1 FeMn	.50	95.0	.25	.10	1.0	-	-	28.0
2 FeMn	2.00	82.5	.35	.05	1.5	-	-	25.0
3 FeMn	.10	82.5	.35	.05	1.25	-	-	38.0
4 FeSi	.15	-	.05	.04	92.5	-	-	24.2
5 FeSi	.15	-	.05	.04	50.0	-	-	20.0
6 FeSi	.50	-	.06	.04	27.5	-	-	17.0
7 FeSi	1.50	-	.15	.06	16.0	-	-	14.0
8 Cr	.10	-	-	-	-	96.0	-	131.0
9 FeCr	.04	-	-	-	1.5	70.0	-	41.0
10 FeCr	1.0	-	-	-	1.5	70.0		37.5
11 Ni	.10	-	-	.02	-	-	99.0	74.2
12 Ni	.25	-	-	.05	-	-	98.9	73.0
13 Ni	.75	-	-	.07	-	-	97.7	71.6
14 Pig	4.42	.13	.019	.026	.70	-	-	3.4
15 FeCrSi	.05	-	-	-	43.5	37.5	-	33.0
16 SiMn	1.5	66.5	-	-	19.0	-	-	14.5
17 Scrap	.22	1.8	.03	.03	.65	25.0	20.0	-
18 Stock	.25	.85	.04	.04	.30	.50	.50	8.5
19 Stock	.13	.50	.025	.025	.30	3.5	1.6	10.5

A matrix of coefficients is constructed, and information from the matrix is encoded in machine language for the computer to operate on. Hand or desk calculator computation could be attempted depending on the size of problem, and for this a simplex tableau would be constructed. For the particular problem under discussion, hand computation is unfeasible.

The machine solution to this problem is shown in Table III which corresponds to Table I, but lists the materials which L. P. says will, when combined, satisfy the requirements and restrictions at least cost. This least cost is $220.88 or about $24 saved per 1000 lbs. ($48,000 per 1000 tons). The final cost is most sensitive to those materials used in greatest quantities. So if several other bar stocks were evaluated in the problem, a lesser-cost solution might well have been found. In general, the more materials evaluated, the more likely are we to find a lower cost solution. There are two reservations to the above statement. First, it is entirely possible that certain materials can be eliminated from consideration by inspection (aside from those materials that will not even do from technological considerations). Second, it is assumed that no extra computational costs are incurred when more materials are evaluated. This assumption is a reasonable one, especially when the computation is by machine.

FIGURE I: EQUATIONS USED IN THE LINEAR PROGRAMMING SOLUTION OF THE LEAST-COST-MIX PROBLEM

A. DEFINITION OF TERMS

Let X_i represent the number of lbs. of input i used in the solution. Subscripts (i's) correspond to the numbers associated with the list of inputs in Table II.

S_{jk} represent slack variables inserted for computational purposes in equation j.

M be a high cost associated with certain artificial slack variables, so that these "artificial" variables will be driven from solution.

R be the total weight requirement.

B. THE EQUATIONS

a) Carbon balance.

$$.50X_1+2.00X_2+.10X_3+.15X_4+.15X_5+.50X_6+1.50X_7+.10X_8+.04X_9+1.0X_{10}+.10X_{11}+.25X_{12}+.75X_{13}+4.42X_{14}+ .05X_{15}+1.5X_{16}+.22X_{17}+.25X_{18}+.13X_{19}+S_{a1}+S_{a2} = .25R$$

b) Manganese balance.

$$95X_1+82.5X_2+82.5X_3+.13X_{14}+66.5X_{16}+1.8X_{17}+.85X_{18}+.50X_{19}+S_b = 2R$$

c) Phosphorus balance.

$$.25X_1+.35X_2+.35X_3+.05X_4+.05X_5+.06X_5+.15X_6+.019X_{14}+.03X_{17}+.04X_{18}+.025X_{19}+S_{c1}+S_{c2} = .03R$$

d) Sulfur balance.

$$.10X_1+.05X_2+.05X_3+.04X_4+.04X_5+.04X_6+.06X_7+.02X_{11}+.05X_{12}+.07X_{13}+.026X_{14}+.03X_{17}+.04X_{18}+.025X_{19}+ S_{d1}+S_{d2} = .03R$$

e) Silicon balance.

$$1.0X_1+1.5X_2+1.25X_3+42.5X_4+50X_5+27.5X_6+16X_7+1.5X_9+1.5X_{10}+.70X_{14}+43.5X_{15}+19X_{16}+.65X_{17}+.30X_{18}+ .30X_{19}+S_e = .75R$$

f) Chrome balance, upper limit.

$$96X_8+70X_9+70X_{10}+37.5X_{15}+25X_{17}+.50X_{18}+3.5X_{19}+S_{f1}+S_{f2} = 26R$$

g) Chrome balance, lower limit.

$$96X_8+70X_9+70X_{10}+37.5X_{15}+25X_{17}+.50X_{18}+3.5X_{19}+S_{g1}+S_{g2} = 24R$$

h) Nickel balance, upper limit.

$$99X_{11}+98.9X_{12}+97.7X_{13}+20X_{17}+.50X_{18}+1.6X_{19}+S_{h1}+S_{h2} = 22R$$

i) Nickel balance, lower limit.

$$99X_{11}+98.9X_{12}+97.7X_{13}+20X_{17}+.50X_{18}+1.6X_{19}+S_{i1}+S_{i2} = 19R$$

j) Equation limiting the scrap used to 30 % of total weight.

$$X_{17} + S_j = 30R$$

k) Total weight balance.

$$X_1+X_2+X_3+X_4+X_5+X_6+X_7+X_8+X_9+X_{10}+X_{11}+X_{12}+X_{13}+X_{14}+X_{15}+X_{16}+X_{17}+X_{18}+X_{19}+S_k = 100R$$

l) The objective function (cost equation)

$$28X_1+25X_2+38X_3+24.2X_4+20X_5+17X_6+14X_7+131X_8+41X_9+37.5X_{10}+74.2X_{11}+73X_{12}+71.6X_{13}3.4X_{14}+33X_{15}+$$
$$14.5X_{16}+8.5X_{18}+10.5X_{19}MS_{a1}+MS_b+MS_{c1}+MS_{d1}+MS\ +MS_{f1}+MS_{g1}+MS_{h1}+MS_{i1} \ldots\ldots\ldots +MS_j+MS_k = \text{MINIMUM.}$$

It is possible to reduce the average material cost per heat still further, although probably not to any great extent. The number of materials utilized in a prescription is limited in the L. P. procedure by the number of equations, or rows in the matrix of coefficients. The number of rows equals material inputs plus slack variables. If one slack variable could be driven out, extra material input could be brought in, which might reduce the overall material cost.[7] A slack can generally be driven out by reducing a requirement coefficient by the amount of the slack which came out of solution, and recomputing. This was done, and the new materials to be charged (8 in all, as against 7 before) are listed in Table IV. The new cost per 1000 lbs. is $219.21. If such a meagre cost saving (compared with the $220.88 prescription) is typical, the extra ordering and handling costs imposed by the extra input would negate the use of this last prescription, unless other considerations prevailed.

TABLE III

NO. CORRESPONDING TO MATERIALS LISTED IN TABLE II	MATERIALS TO STOCK IN INVENTORY	COMPOSITION, % C	Mn	Si	Cr	Ni	COST $/LB.	LBS. REQUIRED IN PRESCRIPTIOI PER 1000 LBS.
1	FeMn	.50	95.0	1.0	-	-	.28	9.3
16	Si-Mn	1.5	66.5	19.0	-	-	.14	3.6
10	FeCr	1.0	-	1.5	70.0	-	.375	64.1
9	FeCr	.04	-	1.5	70.0	-	.41	169.3
11	Ni	.1	-	-	-	99.5	.74	129.0
17	Generated Scrap	.22	1.8	.65	25.0	20.0	-	300.0
18	Stock	.25	.85	.30	.50	.50	.085	324.7

Total cost of 1000 lb. prescriptions: $220.88

It is a fact that there almost always exist materials which are substitutes for others. The L. P. method tests the economic suitability of using such substitutes. For example, in the last prescription a more expensive bar stock (No. 19) was chosen, which because of its rich alloy content, lessened the amounts prescribed of certain ferroalloys. Thus, the final cost was still comparable (in fact less, because of the additional input allowed) despite the use of 341 lbs. of an "expensive" material. It often happens that utilization of the expensive input results in a cheaper overall prescription.

All L. P. computations were accomplished on an I.B.M. 650. A linear programming canned program was available, which made programming of the computer unnecesssary. L. P. routines have been written for several of the available medium and large scale digital computers, and if the foundry does not own or share such a machine, the service bureaus of the larger computer manufacturers will make them available for a fee. Once again, an economic decision must be made before embarking on such a program. However, since for a machine comparable to the I.B.M. 650 the calculation lasts but a few minutes, and will be required several times a year at most, the computational costs are likely to be reasonable.

TABLE IV

NO. CORRESPONDING TO MATERIALS LISTED IN TABLE II	MATERIALS TO STOCK IN INVENTORY	COMPOSITION, % C	Mn	Si	Cr	Ni	COST $/LB.	LBS. REQUIRED IN PRESCRIPTION PER 1000 LBS.
1	FeMn	.50	95.0	1.0	-	-	.28	9.8
16	SiMn	1.5	66.5	19.0	-	-	.14	4.5
10	FeCr	1.0	-	1.5	70.0	-	.375	121.7
9	FeCr	.04	-	1.5	70.0	-	.41	96.9
13	Ni	.75	-	-	-	97.7	.72	7.1
11	Ni	.1	-	-	-	99.5	.74	118.2
17	Generated Scrap	.22	1.8	.65	25.0	20.0	-	300.0
19	Stock	.13	.50	.30	3.5	1.6	.105	341.7

Total cost of 1000 lbs. prescription: $219.21

CONCLUSIONS. Those foundries which have a daily choice of charge inputs and a computer on hand might give some thought to the calculation of charges on a day-by-day basis. Even without a computer, there are successful manual techniques which have been developed, usually incorporating simplifications to ease the computational burden. Descriptions of these techniques can be found in the literature.[6]

The L. P. application described in this paper allows a foundry to make a choice of inputs to stock in inventory. The actual day-by-day charges will generally be calculated separately, unless day-by-day conditions are identical to those incorporated in the L. P. model. For foundries using L. P. to calculate the actual charges, D. E. Debeau[8] points out a limitation in the use of the technique: the composition of each charge input must be known within narrow limits. In those foundries where non-uniform purchased scrap is a major ingredient, L. P. is likely to prescribe a substantial amount of this low-cost scrap, which because of the variability of its composition will result in a relatively high number of off-heats.

A foundry may wish to know the best materials to stock in inventory for a specific production schedule, or to meet a fluctuating demand. In the first instance, the required chemical analyses for the individual company products can be pooled together, and the optimum raw-material mix to be stocked may then be calculated by L. P. The foundry may wish to recompute the optimum raw-material mix whenever there is a cost-change in one of the inputs, when there is a product requirements change, when new materials become available, or periodically. If the foundry employs economic lot-size formuli for the determination of stock quantities, a new prescription calling for substitute materials could not be put into effect until old materials are utilized. Therefore, the foundry might wish to standardize the inventory

cycle* for all charge input materials, and calculate the optimum charge by L. P. at each cycle. ** Whether or not such a standard cycle is used, depends on a cost analysis which needs to be undertaken. In general, the total variable inventory cost for all charge inputs using the standard inventory cycle time, will exceed the total cost when each input is stored at its own optimum level. Before adopting a change in the inventory system, this extra expense must be compared with the estimated savings from Linear Programming prescriptions, and the convenience of ordering all charge materials together.

Since each application is individual, it remains for each foundry to determine if such methods as were described in this paper could be incorporated (inevitably with alterations) into their system to advantage. The systems analysis should be done carefully, lest some significant hidden cost or benefit be overlooked.

ACKNOWLEDGMENT. The cooperation of the I.B.M. Corporation during the computational phase of this program, is gratefully acknowledged.

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*Ordinarily the inventory cycle is different for each material.

**It is assumed that the charge inputs represent a relatively small fraction of the foundries total purchasing volume, so that ordering all 8 inputs at once does not overtax the space available.

AN OPTIMAL METHOD FOR PRODUCT DEVELOPMENT*

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In recent years, there has been a deluge of criticisms, suggestions, and panaceas for the conduct of "research and development" and "product development." The newspapers, scientific and engineering literature, consulting management house organs, and textbook publishers have covered the many facets without analysis of the whole operation. Relatively new technical societies, like the Operations Research Society of America, Society for Advancement of Management, and The Institute for Management Sciences, are concerned with decision-making in general management with only sporadic tackling of the management of the R&D process.

The problem is essentially to develop a product, desired by a customer, by a date dictated by readiness for product utility and by the greatest conservation of resources (men and money). There should be a methodology which can be applied to any product, military or commercial, which will assure the optimal use of manpower (claimed currently to be in short supply) and dollars (also a medium ever to be conserved) so that a product can be put into use at a time commensurate with maximum pay-off.

Try as one may to separate the time factor from allocation of manpower resources and dollars and the reaping of profits or pay-off, the separation cannot be made--as will be demonstrated throughout this exposition. Technological decay and environmental decay and the present worth of future dollars further dictate the consideration of the time factor in evaluating costs and pay-off. To resign ourselves to haphazard, random, or unscheduled conduct of programs (i.e., projects with loose or no completion dates) is to disregard the importance of timing and consequently of technological decay of a current product and useful life of a future product.

There is a serious difficulty in getting across the concept that developments of military and of commercial products do not differ in any way. There is general acceptance today of the "fact" that the former does not involve a profit motive to the organization whereas the latter does. Suppose we now change the phrase "profit motive" to "pay-off value" which is broader in meaning and certainly encompasses the concept of profit taking. This change to the use of broader (and more basic) terminology must be accepted by even the most reticent but still leaves us with the difficulty of relating "pay-off" for both military and commercial products in the same terms, as, for example, in terms of dollars. Even this difficulty can be overcome logically and with utmost meaning and utility.

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* Based on a report "A MODERN DYNAMIC APPROACH TO PRODUCT DEVELOPMENT", Dec 1958, made under the auspices of a Secretary of the Army's Research and Study Fellowship, and to be available from Office of Technical Services, Dept. of Commerce, in July 1959.

Returning to the topic of importance of continual and burgeoning development of new products, it is noted that the impact of "research and development" in a technology changing at an increased rate has taken on national and local importance, often with ominous overtones. Perhaps the most recent expressions of this problem are the paper by Ellis A. Johnson, "The Crisis in Science and Technology and its Effect on Military Development" (Ref. 1) and the paper by J. Sterling Livingston, "Decision Making on Weapons Development" (Ref. 2). One critical point in this picture is the fact that time periods of 7-8 years up to 10 years for the development of a product are unsatisfactory when its useful life might only be 5 years (e.g., for guided missiles). Indeed, a needed product might be obsolescent by the time it is designed, developed, and ready for production or issue to the user or customer.

The problem restated is that for any product in a rapidly changing or competitive technology the history of an item can be longer than desired and that the useful life of the product may not be long enough. Besides <u>time</u>, and the long and short of it, the product to be developed must have a "pay-off" sufficiently high so that we do not lose, along with competitive position, the costs and resources expended in product development. The components of the problem restated are as follows:

1. Pay-off or net profit (p)

2. Useful life or market life (time period for profit taking) (T)

3. Development costs (c)

4. Development time (t)

The decision to acquire or start the development of a new product, or improve a current product, is so important that it can only be conceived of as being the responsibility of top management. Except when top management wishes to pay for prestige, for denial of a market to a competitor, or for other similar intangibles, the decision must be based on a quantitative measure of pay-off. In order to have high levels of precision and accuracy, the estimates going into computation of pay-off should be prepared by the specializing departments who will later be expected to execute the development, production, servicing, and other, programs.

A viewpoint different from that often expressed in discussions or other papers is also necessary concerning the terms "state of the art," "degree of newness," "risk or probability of successful conclusion." It is opinionated that such terms have seriously retarded progressive and dynamic thinking and solution of the evaluation and selection problem.

Some contemplated products are recognized as having very little prior experience to go by in design, production, or use. The "state of the art" is considered to be low and the "degree of nearess" is high. With such an item it is also likely that the "probability of successful conclusion" is considered very low. What do we really mean by such terminology? Do we

not really mean that there is not enough knowledge, facts, and experience for us to feel really confident about the future? Well, similar situations exist to a lesser degree with all but the very simplest developments. The crux of the problem is that all unknown facts and experiences can be obtained at some cost from applied research and pilot plant lots, thorough market analysis, and thorough testing. We are stating that any development can be successful--if we are willing to pay enough in dollars and manpower.

There is now a true meaning to the risk of being unsuccessful. There is a very definite risk of not successfully concluding the development and marketing of a product if we have in any way stinted on funds and personnel necessary to have a thorough market analysis or feasibility study, to acquire missing data and understanding of processes involved, to test out all phases, or for the design and development itself. The risk of success can then be recognized as the ratio of the funds actually supplied to the total funds necessary for successful completion of the product. There can also be a risk of failure in meeting a completion date, no matter how much funding and personnel are furnished; but on the other hand there are examples of very difficult complete developments and production of very new products in as short a time as 18 months to 2 years.

With the statement of a "reasonable" completion date there can be no such claim of a risk of failure--it is only a question of how much do you want to pay to assure a successful development.

Before considering a model for the system, which model is to be optimized for maximum pay-off to the organization, we must discuss how pay-off or profit for a single product in the system can be determined.

In the search for a meaningful, absolute measure of the pay-off or net profit of a proposed product, where maximizing the sum of all the net profits is the prime objective of the firm, the simplest most direct relation that can be found is that of subtracting all the costs chargeable to the product from the gross income. But this is not as useful as separation of the development costs from other costs since we are concerned with the magnitude of development costs and since these costs occur in a different time period from other product costs. Further, we have seen that time has an economic value and that market life is distinct from development life. We thus progress from equation 1 to 2 to 3, as follows:

Eq. 1 $z = P$

Eq. 2 $z = P - C$

Eq. 3 $z = pT - ct$

where z = product value, P = total net profit, C = total development cost, p = average net profit/year, T = market or useful life in years, c = average development cost/year, and t = years of development.

The definitions of the equation 3 factors are stated as they are for ease of computation, fitting in with forecasting and scheduling, and with budgeting procedures. For example, the basis of a year is used in most budgeting and discounting procedures. The profit and cost picture per year is usually desired, as well as the total profit. Further, it is

convenient and possible to express annual costs and profits in terms of the average or arithmetic mean, even though there are variations between years; we thus are considering the same total, i.e., the area under the curve, whether yearly values are constant or variable, but using a square distribution instead of a curvilinear one.

Equation 3 is basic and it is complete as an expression of pay-off and net profit for one product. Every product characteristic and its magnitude contributes to increasing one of the four parameters of the equation. In the final analysis, though, it is the design of the product that establishes the rough location of costs and it is the degree of efficiency of allocation of manpower, time, facilities, and materials that pinpoints the exact cost. It can be seen that in early evaluation of a product's value, where the design or designs are only generally conceived, only rough locations of the pay-off are possible. As progress is made in development and a design becomes more firm, the optimization of later use of facilities, manpower, and materials can be studied leading to closer and closer pin-pointing of costs as well as pay-off. The formula can, and should be, used for periodic re-evaluations of the product value.

The economic value of time can be accounted for by discounting future dollars for their present value. Since costs and profits can be stretched out in time by years, discounting can be applied by multiplying dollars by $(1 + i)^{-t}$ or its approximation, e^{-it}. However, two areas of time are still not accounted for, one being the desire to maximize both "short-range" and "long-range" profits and the other being the problem of evaluating pay-off whether in a peace-time or war-time economy. If our formula (Equation 3) can be modified to take care of these two "time" areas it has then become even more basic and useful. An attempt to do this follows.

Getting "short-range" profits or benefits is not necessarily incompatible with maximizing "long-range" profits. From the short-range viewpoint we want to recognize and reward those projects which have shorter than average development times so that profit-taking can start earlier than it would have if we had a project of average development time. (For any class or type of products, managers and historians will recognize the existence of an average development time expected for that class or type, e.g., recent figures indicate 10 years development for a guided missile, 5 years for a new electrical appliance, 3 years for a major modification of an automobile.) We must also recognize that development time can be shortened or lengthened almost at will by the demand of the customer, user, the director of the New Products Department, or the Sales or Logistics Department; changing of the expected delivery date would also be expected to affect the cost of development.

The relation then is that a development time, t, equal to that of the average, $\bar{t}$, does not affect our profit picture, but if $t < \bar{t}$ there is an ancrease in total net profit, $p\bar{T}$, and if $t > \bar{t}$ there is a loss of profit. Figure 1A illustrates this relationship when $\bar{t} = 3$ years. Equation 4 also represents this relationship (assumed linear).

Eq. 4 $\Delta(p\bar{T}) = (-p\bar{T}/\bar{t})t + p\bar{T} = p\bar{T}\,(1 - t/\bar{t})$

Note that $\Delta(p\overline{T}) = 0$ (no change in $p\overline{T}$, total net profit) when $t = \overline{t}$. Also, $\Delta(p\overline{T}) = -p\overline{T}$ (the loss of net profit of one whole average project) when development time is twice as long as that of the average project, i.e., when $t = 2\overline{t}$.

From the long-range viewpoint, we would prefer products that have a market life longer than that of an average product of the same type or class. In fact, if a product has a useful, profit-making, or market life that is twice as long as the average it has in fact saved us the necessity and cost of one whole development project. Thus, if a product will last 6 years instead of the average of 3 years which is known to exist for similar products, there is no need for a development program to provide a product to cover the profit-taking of the last 3 years of the 6. Figure 1B illustrates this relationship and is represented by Equation 5.

$$\text{Eq. 5} \quad \Delta(c\overline{t}) = (c\overline{t}/\overline{T})T - c\overline{t} = c\overline{t}\left[(T/\overline{T}) - 1\right]$$

Adding the $\Delta(p\overline{T})$ and $\Delta(c\overline{t})$ of Equations (4) and (5) to Equation (3), we get:

$$\text{Eq. 6} \quad z = pT + p\overline{T}\,(1 - t/\overline{t}) - ct + c\overline{t}\left[(T/\overline{T}) - 1\right]$$

and simplifying by rearrangement of terms:

$$\text{Eq. 7} \quad z = p\left[T + \overline{T}(1 - t/\overline{t})\right] - c\left[t + \overline{t}\,(1 - T/\overline{T})\right]$$

If $T^* = \left[T + \overline{T}\,(1 - t/\overline{t})\right]$

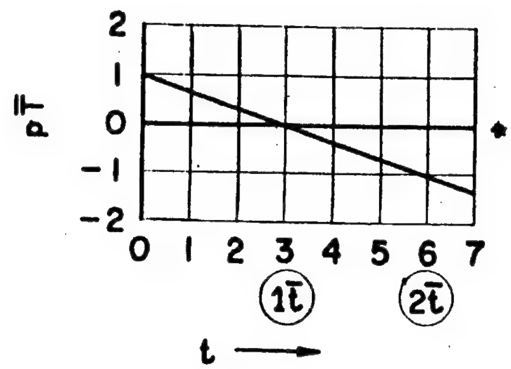
and $t^* = \left[t + \overline{t}\,(1 - T/\overline{T})\right]$

then, when $T \neq \overline{T}$ and $t \neq \overline{t}$:

$$\text{Eq. 8} \quad z = pT^* - ct^*$$

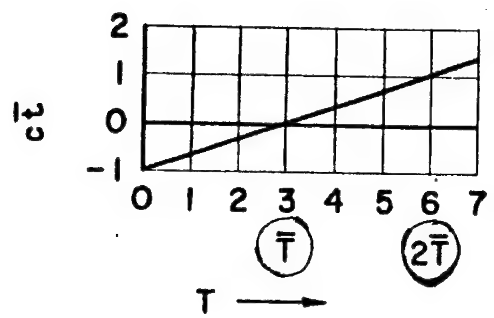
When evaluating programs <u>periodically</u>, using our recommended formula, after development has been started, the value for "t" is that of the <u>remaining years required for completion</u>. Any previous year's development costs are thus excluded from the evaluation of future product worth. This is considered sound in that we are always considering <u>the present worth</u> of the product's future, regardless of the past. To consider the inclusion of past years and their development costs is in the direction of reduction of present worth and may unfairly jeopardize a program in competition with newly conceived products. Retrospection will show that this method actually protects our past expenditures already invested in the development program—but without the need for considering how much time or costs were spent; our only interest is the present worth of future pay-off.

Concerning our previously stated problem of application to both peacetime and war-time situations, the following is quoted from A.S. Locke (Ref 3): "The unit by which cost is measured is different in a peacetime economy from that in a wartime economy. When a nation is at war, the determining unit is that of time. If a known threat exists, a counter weapon must be put into action against it in the shortest possible time. The cost in dollars becomes unimportant; the cost in time becomes all important. In peacetime, the cost of the weapon program in units of dollars becomes the governing factor between competitive weapon systems."



$$\Delta(p\bar{T}) = (-p\bar{T}/\bar{t})t + p\bar{T} = p\bar{T}(1 - t/\bar{t}).$$

A. RELATION OF DEVELOPMENT TIME AND ANNUAL PROFIT.



$$\Delta(c\bar{t}) = (c\bar{t}/\bar{T})T - c\bar{t} = c\bar{t}\left[(T/\bar{T}) - 1\right]$$

B. RELATION OF MARKET LIFE AND ANNUAL DEVELOPMENT COST.

FIGURE 1.

TIME, COST, AND PROFIT RELATIONSHIPS.

By properly relating pay-off, costs, and time, as in Equation 8, we have given a very short time of development, most desirable in wartime (and not really any less, in peacetime), such effect in increasing profit (by the low value of $t/\bar{t}$ in T*) that the "cost in time" overshadows the development costs. This writer does not therefore consider that wartime or peacetime causes a change in how we measure costs but that we forcibly put into effect in wartime measures, such as short development times, that would also have increased pay-off if it had been peacetime.

To illustrate the basic nature of our development of Equation 8 as a measure of pay-off or importance of a product, we refer to Equation 6 which is rewritten as follows:

Eq. 9 $$z = pT + p\bar{T}\,(1 - t/\bar{t}) - ct - c\bar{t}\,(1 - T/\bar{T})$$

For small values of $t/\bar{t}$ and $T/\bar{T}$, we can write $(1 - t/\bar{t})$ as $e^{-t/\bar{t}}$ and $(1 - T/\bar{T})$ as $e^{-T/\bar{T}}$. Thus, we get:

Eq. 10 $$z = pT + p\bar{T} \circ e^{-t/\bar{t}} - ct - c\bar{t} \circ e^{-T/\bar{T}}$$

By comparison with a physical system for high polymers as given by T. Alfrey (Ref 4), where $S = S_o\, e^{-\theta/r}$, θ is the experimental timescale or actual time of application of a stress, S, whose initial value is S_o and r is "thought of as an internal timescale, characteristic of the relaxation of the material." Thus, by comparison our $\bar{t}$ and $\bar{T}$ are averages characteristic of the class or type of product and our t and T are the experimental or estimated development times and product lifetimes. (One should not attempt to substitute actual values in Equation 10 without a correction factor.)

A sensitivity analysis for the errors of estimating should be made for Equation 8. The method for doing this, and the results for one example, are presented below.

The parameters in Equation 8 are engineering and sales engineering or market analysis estimates. The chances are that the profit and cost errors occur at random and in estimating may as often cancel each other as augment each other. Further, estimating errors may tend to cancel each other between projects as well as within projects for any one program. However, an estimate of the maximum errors and the expected dispersion of values due to estimating errors will point out the degree of accuracy needed.

Using $z = pT - ct$, when $T = \bar{T}$, $t = \bar{t}$, we find that:

Eq. 11 $$z + \Delta z = (p + \Delta p)\,(T + \Delta T) - (c + \Delta c)\,(t + \Delta t)$$

and by combining terms, by making all terms positive for maximum deviation, by dropping $\Delta p \circ \Delta T$ and $\Delta c \circ \Delta t$ terms as very small, and by converting to "$\Delta x/x$" fraction equivalents of percentage error, we get:

Eq. 12 $$\text{Max. } \Delta z \cong pT\left(\frac{\Delta T}{T} + \frac{\Delta p}{p}\right) + ct\left(\frac{\Delta t}{t} + \frac{\Delta c}{c}\right)$$

Let p = \$450,000/year, and T = 3 years, pT = \$1,350,000; and c = \$150,000/year and t = 3 years, then ct = \$450,000.

The error in estimating years of development and useful or market years is expected to be small, particularly if estimates and planning are done carefully, and let us say that these are not more than 5% in error. For development costs, it is believed that these can be estimated reasonably with a 10% error provided optimism and enthusiasm does not make one forget the usual and expected need to redesign and retest one or more times. More serious errors are expected in estimating the expected yearly profit and this perhaps would be a 20% error. Thus:

$$\Delta p/p = 0.2$$
$$\Delta c/c = 0.1$$
$$\Delta t/t \text{ and } \Delta T/T = 0.05$$

Substituting values in Equation 12, we get:

$$\text{Max. } \Delta z = 1{,}350{,}000\ (0.25) + 450{,}000\ (0.15)$$
$$= 337{,}500 + 67{,}500$$
$$= 405{,}000$$

$405{,}000/(1{,}350{,}000 + 450{,}000) = 0.225$, or 22.5% for maximum error in z, the product value.

Since several distributions are added, we can expect the distribution for the error in z to be normally distributed and 0.225 corresponds to the 3 σ limit. The $\sigma = 0.075$ and the Probable Error is 0.6745 (0.075) or 0.051. Thus we can expect that in about 68% of our estimates z may have an error of 7.5% or less and half of our estimates will have errors of 5.1% or less.

There is another important concept, useful in management control, that can be gleaned from our basic relationship. This concept concerns "maturity" of a class or type of products. Suppose we rewrite Equation 7 as follows:

$$\text{Eq. 13} \qquad z = pT + p\overline{T} - pt\ (\overline{T}/\overline{t}) - ct - c\overline{t} + cT/(\overline{T}/\overline{t})$$

If $k = (\overline{T}/\overline{t})$, then the larger this ratio of useful life to development the greater the product value, z, for future average projects because of both increased profit and decreased development cost. The factor, k, then is a measure of the "maturity," or position on the growth curve so characteristic of the learning process, aging of animal life, growth of populations, and radioactive transmutations (see Feller, Ref 5). For a mature product, k is high as for hand grenades having an approximate useful life of 25 years with development in the area of 5 years, k = 20/5 = 4. A guided missile, on the other hand, has the expected life of 5.0 years but takes 10 years to develop, hence k = 0.5. The product class of nylon might be expected to have a low value of maturity but because of the thorough basic research and the development by Carothers and his duPont staff it actually has a high maturity; we take the liberty of estimating that its life will go another 10 years, hence a total of 30 years (1939 - 1969) whereas development (excluding the basic superpolyamide research) might be of the order of 7 years (1932 - 1939), hence k = 30/7 = 4.3. For the human species, by way of comparison, development might be accepted by most as 18 years and useful life, from 18 to 65; hence, k = 47/18 = 2.6.

The thought now occurs that there is cost associated with and a pay-off resulting from an increase in development time which results in a longer useful life for a product. The useful life of a product is a direct result of its characteristics and design insofar as how thoroughly it satisfies a market or customer's need. The thoroughness and development costs put into the development of nylon is still paying off. The same thoroughness and high costs were necessary for the very existence of a guided missile and gains in reduction of development time and costs can be expected in the near future for any new guided missile required by "market" demands. Using the analogy of human development, a delay in starting one's useful life at 18, as by going to college, increases both pay-off during the working life of an individual and may even extend his working years beyond 65 (if he and his employer so desire). Further, the realization of the problem and the ability of an agency or person to design a more useful and longer-lasting product together with its knowledge of scheduling which leads to reduced development times contributes to higher values of k. Does this not lead to a measure of the "efficience" or "maturity" of that agency or person?

In order to determine the maximum product value, z_{max}, we take the partial differential of z in Equation 13 with respect to profits and separately with respect to costs, and proceed as follows:

Eq. 14 $$\frac{\partial z}{\partial p} = T + \bar{T} - \frac{t\bar{T}}{\bar{t}} = 0; \quad T = \frac{\bar{T}(t - \bar{t})}{\bar{t}}$$

Similarly for costs:

Eq. 15 $$\frac{\partial z}{\partial c} = -t - \bar{t} + \frac{T\bar{t}}{\bar{T}} = 0; \quad T = \frac{\bar{T}(t + \bar{t})}{\bar{t}}$$

From Equations 14 and 15 we get the expected results that the optimal value of $\bar{t} = 0$ and the optimal value of $T = (\bar{T}/\bar{t}) \cdot t = k \cdot t$ (i.e., when $\bar{t} = 0$). Using these two values of $\bar{t}$ and $\bar{T}$ in Equation 13, we get the simple and significant result that $z_{max} = p\bar{T}$. Thus, if the characteristic of a product is that it requires no development time, that is, can be created instantaneously like Adam, or a god in Greek mythology, or by immediate purchase, company merger, or similar means, only then can we get the maximum pay-off or profit. In such a case, the cost of purchase or investment would have to become a cost against the calculation of p, the annual net profit; the discounted cash-flow method analysis or similar investment analysis might then be applicable. It is also interesting to note that for a proven or well-established ("standard") value of k (or $\bar{T}/\bar{t}$) for a class of products, we would do best by having T/t, the estimated ratio of product life to development time for a newly proposed product under evaluation, as equal to k. But we can do better by investing in the improvement of the k value for the whole class of products by advancing the "state of the art" or "maturity" through applied research and/or improvement of the methods of scheduling and allocation of development resources.

When $\bar{t} \neq 0$, but we still have $T = k \cdot t$, the maximum value of z becomes:

Eq. 16 $z = p\bar{T} - c\bar{t}$

or rewritten for plotting on a graph as in Figure 2:

Eq. 17 $\frac{z}{c\bar{t}} = \frac{\bar{T}}{\bar{t}} \cdot \frac{p}{c} - 1$

where $\frac{\bar{T}}{t} = k$ (slope of the line). Figure 2 is plotted for the condition of $k = 1$. We now want to know the value of improving the "maturity" of class of products from $k = 1$ to $k = 2$, keeping the same ratio of the annual average profit to annual average cost, in this case $p/c = 2$. The expected improvement to do this would be expected to be the reduction of one-half the cost of any future whole new product development project. The higher the ratio of profits to costs or the higher we wish to advance the "maturity" of a class of products, the more beneficial does this activity become. The cost of improving "maturity" is the separate estimate of the necessary applied research weighed against the value of improvement for one or more expected new products in the same class or type.

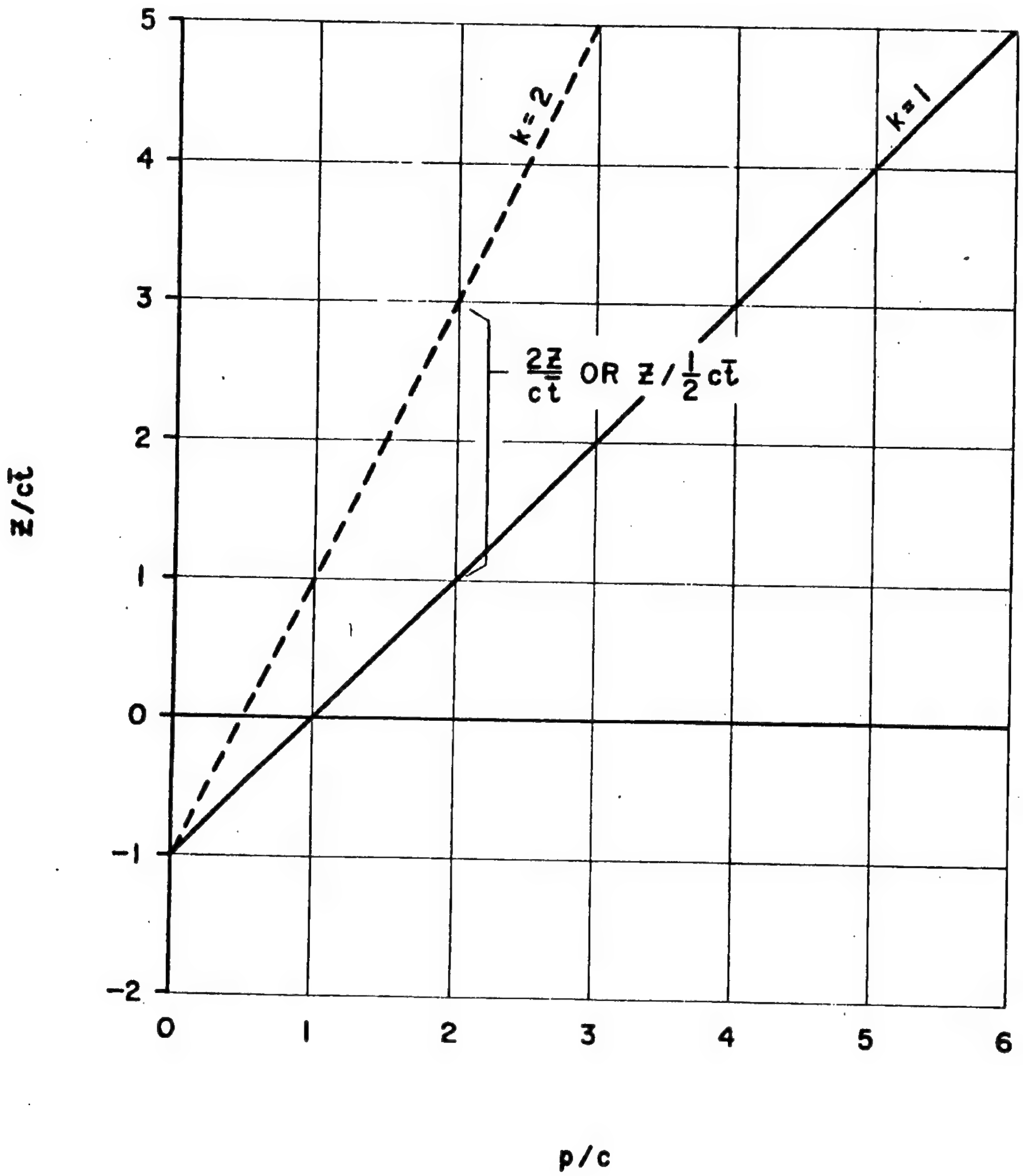
For a commercial product, the use of dollars makes it comparatively simple to compute the gross profit or pay-off and subtract from it all the costs to obtain the net profit in dollars. The problem is now to do the same for the pay-off for a military product, which is often expressed in terms of fire-power, probability of kill, percentage improvement over a competitive existing product, effect on time reduction of a major war, ability to defeat a specific target, and so forth. Unless this can be done, there appears to be little hope of ever obtaining an absolute measure of worth for a military product.

Let us attempt to establish some basic relations for all military products, as follows:

1. The purpose of such products is their possible use in a major war and evaluation of net pay-off should be made under such circumstances. This does not preclude their use or make them less valuable in limited wars or police actions. Once net pay-off is determined, it then is the value to be used as the "p" value in our formulas.

2. There is a cost of getting closer physically to the enemy or an enemy target in order to capture or destroy the same. This must be recognized to relate the costs and pay-offs of weapons having the same effectiveness but at different ranges.

3. All military products can be and should be evaluated ultimately in terms of the major objective of defense of the nation and all its resources, of the military forces (actually a resource in its broadest sense), and of any segment of the military forces down to the individual soldier. The primary purpose of an offensive action and of the ultimate need to win is the <u>defense</u> of the nation, its way of life, and what it stands for.



THE DETERMINATION OF VALUE OF IMPROVING PRODUCT "MATURITY".

Figure 2

4. The basic unit requiring defense in the war effort is the individual soldier. As distasteful as it must be to all, a value in dollars must be placed on this soldier under combat conditions and, based on an estimate of cost of conversion from a civilian to a trained combatant, let us state this will be $8,000. For other units, such as a factory, steel mill, industrial city, we will again have to be heartless and calculating and use a dollar sign measure of its contribution to defense, let us say, in terms of days of sustained combat, for which we can get a cost measure.

An example will illustrate the possibilities of establishing dollar values of pay-off. Hypothetical values only will be used, as well as rounded-off numbers for ease of computation. Let us take the pay-off for a rifle. If a war of atomic warhead long-range ballistic and guided missiles should come to pass, it is possible to so devastate a country that rifles will be of little or no value. However, with opposing nations having retaliatory abilities, it is opinionated that it is not likely that this type of warfare will be tried except as a last desperate effort. So we will then consider that the rifle has a purpose. We will consider that 3 million semi-automatic rifles will be manufactured and 2 million will be issued to troops. Cost of a rifle being set at $100 each, the cost is 3×10^9 dollars. From our logistics people and field commanders, let us say that we find that because of our infantry being equipped with rifles, we can save 300,000 GI's in a war of 3 years' duration. Hence, the gross pay-off is $(8 \times 10^3)(3 \times 10^5) = 2.4 \times 10^9$ dollars against which we have a cost of production only, of 3×10^9. This represents a loss, or an absence of a <u>net</u> pay-off. We are accomplishing the defense objective of saving of our own lives but not very efficiently. Suppose, however, we contemplate the development of an <u>automatic</u> design to replace the <u>semi-automatic</u> rifle, and there is an estimate that we can now save 380,000 lives. The gross pay-off is $(8 \times 10^3)(3.8 \times 10^5) = 3.04 \times 10^9$, with a net profit of 4×10^7 dollars (considering production costs only). In computing the total product value by $z = pT - ct$, let us remember that there is a development cost to be subtracted from the total net pay-off. The product value might be something like the following (including discounting for present worth of future dollars to get the discounted average annual p and c):

$$z = pT\,(1 + a)^{-\left[(T/2) + t\right]} - ct\,(1 + a)^{t/2}, \text{ where } T = \overline{T} \text{ and } t = \overline{t}$$

$$= (4 \times 10^7)(20)(1.03)^{-16} - (5 \times 10^5)(6)(1.03)^3$$

$$= (80 \times 10^7)/1.61 - (30 \times 10^5)(1.09)$$

$$= 497 \times 10^6 - 3.27 \times 10^6$$

$$= 4.94 \times 10^8 \text{ dollars}$$

Time and space cannot be devoted to the methods of computing lives saved, the cost in terms of one day less of war, the value in defense dollars of an industrial plant, and so forth, since each example will be an exercise in itself with innumerable estimates and considerations of probabilities which the logistics and field commanders know best. It can only be indicated that

we have a total economy to defend, whether by defensive or offensive means in varying degrees. The values of alternate approaches and products for this defense can be analyzed in terms of dollars, if we really desire to do so, by the use of ingenuity, imagination, and logic. However, the value of products, commercial or military, in terms of personal satisfaction, political aspirations, cultural gains, protection of a way of life, good will, etc., cannot be evaluated, unless the reader wishes to put a dollar value on any of these "intangibles," or on their impact on an economy.

The preceding discussed a recommended means for determining the value of a product based on estimates of annual net pay-off, product life, development cost, and development time. In no way did we prove that this will give us the optimal method for maximizing profit (or, on the other side of the coin, minimizing costs) to the firm over either or both its short-term or long-range existence, that is, that the method meets the basic objective of the firm. So far we have discussed how one program is evaluated; we must now consider selecting those programs out of the set of all possible programs which will maximize profits over the life of the firm.

Hertz and Feeney (Ref 6) have prepared a basic analysis of an organized industrial "operation" which is also basic to this study. The development (and proof) of the recommended procedure of this study will follow along similar lines but with modifications primarily to include additional values of time itself and of the separation of development costs. Value depreciation over time is also handled as a separation operation (i.e., not included in the equations).

Any organized "industrial" operation is a system concerned with transforming a set of elements, called inputs, into some other set, called a set of outputs, in pursuit of its objective or objectives. An operation is the function carrying one set of elements into another and will be designated as F( ). Our inputs are such things as labor, information, material, etc., which measure as dollar costs and our input set of costs for the i-th product for the period of time j is:

Eq. 18 $$C_{ij} \equiv (c^*_{i1}, c^*_{i2}, \ldots\ldots\ldots, c^*_{in}) \equiv \sum_{j=o}^{t_i} c^*_{ij}$$

Output, or gross profit, P, measured in dollars, is a set of measures produced by the operation during some time interval, thus with z as a remainder:

Eq. 19 $$F(C_{ij}) \equiv P_{ij} - z_i \equiv (p^*_{i1}, p^*_{i2}, \ldots\ldots, p^*_{in}) - z_i \equiv \left(\sum_{j=o}^{T_i} p^*_{ij}\right) - z_i$$

We will define a term net profit as the difference between total gross profit and total costs, so that the worth, z_i, of the i-th product which is:

Eq. 20 $$z_i = \sum_{j=o}^{T_i} p^*_{ij} - F(C_{ij})$$

becomes by this operation on the costs:

Eq. 21 $z_i = \sum_{j=o}^{T_i} p_{ij}$.

Examination of all of the costs indicates that some costs, namely the ones associated with development of the product occur in an independent time sequence, of length t_i, from that of the product time sequence, of length T_i. We therefore can take the liberty of separating these out without change in the relationship, thus:

Eq. 22 $$z_i = \sum_{j=t_i}^{T_i+t_i} p_{ij} - \sum_{j=1}^{t_j} c_{ij} = \sum_{j=1}^{T_i+t_i} (p_{ij} - c_{ij})$$

Another simplification is to define p_{ij} and c_{ij} as the average value over its time period and multiply by the length of its lifetime, thus for any length of time $n \geqq T + t$

Eq. 23 $$z_i = (p_{i.}T_i - c_{i.}t_i)_{j=1}^{n}$$

The objective of the firm is to maximize all products over all time, both short-term and long-range, and this is designated as follows:

Eq. 24 Maximize $$Z = \sum_{i=1}^{m} (p_{i.}T_i - c_{i.}t_i)_{j=1}^{n}$$

subject to any specified restrictions.

The matrix represented by Equation 24 is shown in Figure 3.

Fig 3. Matrix of profits and development costs over time

		TIME $j=1$ ------------ n							Sums over j
	$i=1$	$-c_{11}$	$-c_{12}$	----	$-c_{1t}$	$p_{1,t+1}$	---	p_{1n}	$p_{1.}T_1-c_{1.}t_1$
		$-c_{21}$	$-c_{22}$	---	$-c_{2t}$	$p_{2,t+1}$	---	p_{2n}	$p_{2.}T_2-c_{2.}t_2$
PRODUCTS		---	---	$-c_{ij}$	---	---	p_{ij}	---	$p_{i.}T_i-c_{i.}t_i$
	m	$-c_{m1}$	$-c_{m2}$	-----	$-c_{mt}$	$p_{m,t+1}$	--	p_{mn}	$p_{m.}T_m-c_{m.}t_m$
						Sum over j and i =			$\sum_{i=1}^{m}(p_{i.}T_i-c_{i.}t_i)_{j=1}^{n}$

For the i-th product, the development costs, $-c_{i1}$, $-c_{i2}$, ----, $-c_{it_i}$, form a basis, and the profits, p_{i,t_i+1}, p_{i,t_i+2}, ----, p_{i,T_i}, form another basis, but both belong to the same set of transformed inputs (i.e., outputs) for the i-th product vector.

It is not clear how we can maximize Z for the firm until an investigation is made of the most elemental matrix that can be formed, then of the elemental matrix expanded by 1 in the j-th direction, and then by 2, 3, -----, ∞. We can then repeat this procedure for the i-th direction, and then generalize for m products over n years for any firm.

Suppose we consider the most elemental system where we want at least one "-c" in every i-th row and only one "p" in every j-th column <u>after the j=1 column</u>. In the entire matrix, we will simplify life by making all c's equal and all p's equal. By assuming the above, we are saying that there must be a profit, p, in each year after year 1, and that there must be a development, and its cost, c, for every product.

The most elemental matrix, Q_{22}, is:

$$Q_{22}: \quad \begin{array}{|c|c|} \hline -c & p \\ \hline o & o \\ \hline \end{array} \qquad \begin{array}{c} p - c \\ o - o \\ \hline Z = p - c \end{array}$$

Expanding Q_{22} by 1 in the j-th direction, we now find three possibilities, Q^1_{23}, Q^2_{23}, and Q^3_{23}:

$$Q^1_{23} \quad \begin{array}{|c|c|c|} \hline -c & p & o \\ \hline o & -c & p \\ \hline \end{array} \qquad \begin{array}{c} p - c \\ p - c \\ \hline Z = 2p - 2c \end{array}$$

$$Q^2_{23}: \quad \begin{array}{|c|c|c|} \hline -c & p & p \\ \hline o & o & o \\ \hline \end{array} \qquad \begin{array}{c} 2p - c \\ o - o \\ \hline Z = 2p - c \end{array}$$

$$Q^3_{23}. \quad \begin{array}{|c|c|c|} \hline -c & -c & p \\ \hline -c & p & o \\ \hline \end{array} \qquad \begin{array}{c} p - 2c \\ p - c \\ \hline Z = 2p - 3c \end{array}$$

As far as the value of Z is concerned, we can conclude that $Q^2_{23} > Q^1_{23} > Q^3_{23}$, specifically because $-1c > -2c > -3c$.

Expanding Q_{23} by 1 in the j-th direction, we find for all possibilities (totalling 5) meeting our basic assumptions:

Q^1_{24}:

-c	p	o	o
o	-c	p	p

$$\begin{array}{r} p - c \\ 2p - c \\ \hline Z = 3p - 2c \end{array}$$

Q^2_{24}:

-c	p	p	o
o	o	-c	p

$$\begin{array}{r} 2p - c \\ p - c \\ \hline Z = 3p - 2c \end{array}$$

Q^3_{24}:

-c	p	p	p
o	o	o	o

$$\begin{array}{r} 3p - c \\ o - o \\ \hline Z = 3p - c \end{array}$$

Q^4_{24}:

-c	-c	p	p
-c	p	o	o

$$\begin{array}{r} 2p - 2c \\ p - c \\ \hline Z = 3p - 3c \end{array}$$

Q^5_{24}:

-c	-c	-c	p
-c	p	p	o

$$\begin{array}{r} p - 3c \\ 2p - c \\ \hline Z = 3p - 4c \end{array}$$

We now find that: $Q^3_{24} > Q^1_{24}$ <u>or</u> $Q^2_{24} > Q^4_{24} > Q^5_{24}$. If our costs <u>increase over time</u> by $(1+a)^{t-1}$ after the first year we can then also evaluate the 1st and 2nd matrices as $Q^1_{24} > Q^2_{24}$.

It should be clear now that expanding by one step at a time until m years, the same results will be obtained that <u>the best</u> pay-off is obtained by $Q^{Max.}_{2m}$:

$Q^{Max.}_{2n}$:

1	2		n
-c	p	-----	p
o	o	-----	o

$$\begin{array}{r} (n-1)\ p - 1c \\ o - o \\ \hline Z_{max} = p(n-1) - c(1) \end{array}$$

The worst pay-off is obtained when a program exists where $z_1 = p(1) - c(n-1)$ (e.g., see Q^5_{24} and Q^3_{23}), necessitating an additional program, to take care of supplying one and only one "p" to the remaining years beyond the first year, i.e. requiring a program where $z_2 = p(n-2) - c(1)$. Thus:

$$Z_{min} = z_1 + z_2 = p(1 + n - 2) - c(n - 1 + 1)$$
$$= p(n-1) - c(n)$$

Matrix Q^{Min}_{2n} represents the <u>worst</u> pay-off

$$Q^{Min}_{2n} \qquad \begin{array}{ccccc} 1 & 2 & & n-1 & n \\ \hline -c & -c & ----- & -c & p \\ -c & p & ----- & p & o \end{array} \qquad \begin{array}{l} p(1) - c(n-1) \\ \underline{p(n-2) - c(1)} \end{array}$$

$$Z_{Min} = p(n-1) - c(n)$$
$$= p(n-1) - c(2n-3)$$

Let us now extend the Q_{24} matrix by 1 in the i-th direction. The nine Q_{34} possibilities are:

Q^1_{34}:

-c	p	o	o	p - c
o	-c	p	o	p - c
o	o	-c	p	p - c

$Z = 3p - 3c$

Q^2_{34}:

-c	p	o	o	p - c
o	-c	p	p	2p - c
o	o	o	o	o - o

$Z = 3p - 2c$

Q^3_{34}:

-c	p	p	o	2p - c
o	o	-c	p	p - c
o	o	o	o	o - o

$Z = 3p - 2c$

Q^4_{34}:

-c	p	p	p	3p - C
o	o	o	o	o - o
o	o	o	o	o - o

$Z = 3p - c$

Q^5_{34}:

−c	−c	p	o	p − 2c
−c	p	o	o	p − c
o	o	−c	p	p − c

$Z = 3p - 4c$

Q^6_{34}:

−c	−c	p	p	2p − 2c
−c	p	o	o	p − c
o	o	o	o	o − o

$Z = 3p - 3c$

Q^7_{34}:

−c	−c	p	o	p − 2c
o	o	−c	p	p − c
−c	p	o	o	p − c

$Z = 3p - 4c$

Q^8_{34}:

−c	−c	−c	p	p − 3c
−c	p	o	o	p − c
o	−c	p	o	p − c

$Z = 3p - 5c$

Q^9_{34}:

−c	−c	−c	p	p − 3c
o	−c	p	o	p − c
−c	p	o	o	p − c

$Z = 3p - 5c$

Considering future value of costs, by the factor $(1+a)^{t-1}$, we get the following:

$$Q^4_{34} > Q^2_{34} > Q^3_{34} > Q^6_{34} > Q^1_{34} > Q^5_{34} \text{ or } Q^7_{34} > Q^8_{34} \text{ or } Q^9_{34}$$

Again we come to the same conclusions as before, i.e., that the program represented by Q^4_{34} is the best—maximizes total profit over time to the firm and the first line of Q^8_{34} or Q^9_{34} will give us the worst profit picture.

Extending the picture step-wise in the i-th direction until m projects are reached, we still have the Q_{2n}^{Max} matrix as the best, but repeating to show m projects over n years:

Q_{mn}^{Max}:

	1	2	-----	n	
1	-c	p	-----	p	p(n-1) - c(1)
2	o	o		o	o - o
|	|	|		|	
m	o	o	-----	o	o - o

$Z_{max} = p(n-1) - c(1)$
regardless of the magnitude of $m(m \geqq 1)$

The worst picture is shown by Q_{mn}^{Min}:

Q_{mn}^{Min} :

	1	2	3	--------	n-2	n-1	n	
1	-c	-c	-c	-c	-c	-c	p	p(1) - c(n-1)
2	-c	p	o	--o--	o	o	o	
3	o	-c	p	--o--	o	o	o	
|	|	|	|	-c,p,o	o,p	o	o	p(n-2) - c(m-1)
m	o	o	o	o	-c	p	o	

$$\begin{aligned} Z_{Min} &= p(1)+p(n-2)-c(n-1) \\ &\quad -c(m-1) \\ &= p(n-1)-c(n+m-2) \end{aligned}$$
where $m \leqq n-1$

when $m=n-1$, $Z_{min} = p(n-1)-c(2n-3)$

The preceding illustrations leading to the development of maximum pay-off of $Z_{max} = p(n-1) - c(1)$ should be recognized as manipulation of the number of p's and c's as variables so that all possibilities could be examined and understood. However, given a proposed program, with fixed estimates of product life and development time, how shall we calculate little z_i for the project so that the sum of all z_i's will give us the Z_{max} for the whole program? We have already derived the strategy as shown by equations 7 and 8 and figure 1.

In order to be rigorous, we will use our profit and cost matrices, and the results derived from them, for developing the best product strategy. We must first designate actual times for the i-th project as T_i for the number of years of profit taking where $T_i \leqq n-1$ and as t_i for the number of years during which the development costs are incurred where t_i also is $\leqq n-1$. $T_i + t_i \leqq n$.

It appears inevitable that we must recognize as a reference point the inherent measure of central tendency of time within the set of all similar products. This was discussed with reference to the physical phenomena of Alfrey's internal timescale (relaxation time), r, where the experimental timescale was ⓣ and the original property of the material varied by the factor $e^{-ⓣ/r}$. We similarly recognize an internal time scale as the average product life, $\bar{T}$, and the average development time, $\bar{t}$, for a class of related products. Historical data, or at least comparisons, supply the values for $\bar{T}$ and $\bar{t}$.

Let us use the elemental matrices to derive the optimal strategy, and the arbitrarily assigned values of $\bar{T} = 1$ and $\bar{t} = 1$. For a matrix Q_{12} or Q_{22} (see above), we get $Z = z_i = p\,\bar{T} - c\,\bar{t} = p\,T - c\,t$. In Q^2_{23} (also see above), $T = 2\,\bar{T}$ and results in the saving of another development project and its cost, c, over the situation where projects of average product life and development time would have existed, as in Q^1_{23}. In Q^3_{34}, we again experience the saving of one development cost and in Q^4_{34}, we save two development costs when $T = 3\,\bar{T}$. Thus by further extension to the Q_{mn} matrix, we have proven that when $T = n\,\bar{T}$ we save (n-1) c costs. $n = T/\bar{T}$ so that the saving is $(T/\bar{T}-1)\cdot c$ over the situation where only average projects exist. Further, let us not forget that the number 1, for one c, must be inserted, or instead we should substitute its true symbol, $\bar{t}$, thus we get that the saving is $c\ (1)\ (T/\bar{T} - 1)$ or $c\ \bar{t}\ (T/\bar{T} - 1)$. By way of further explanation, we will now say that the normal or average project has $\bar{t} = 2$ and $\bar{T} = 1$. The normal matrix is then Q^6_{24} and the actual case, Q^7_{24} as follows:

Q^6_{24}:

-c	-c	p	o	p - 2c
o	-c	-c	p	p - 2c

$Z = 2p - 4c$

Q^7_{24}:

-c	-c	p	p	2p - 2c
o	o	o	o	o - o

$Z = 2p - 2c$

We will note that $c\ \bar{t}\ (T/\bar{T}_7 - 1) = c(2)\ (2 - 1) = 2c$ which is borne out by comparison of Q^6_{24} and Q^7_{24}.

Proof of the extension to the n-th case for any $\bar{T}$ and T is obvious. Further proof that the increase of the value of p due to decrease of development time below the average time $\bar{t}$ is $p\bar{T}\,(1 - t/\bar{t})$ would merely be an exercise.

Thus, where the average project value is:

$$\text{Eq. 25} \qquad z_i = p_i \cdot T - c_i \cdot t = p_i \cdot \bar{T} - c_i \cdot \bar{t}$$

the extended formula for any T or t, given any $\bar{T}$ and $\bar{t}$ is:

$$\text{Eq. 26} \qquad z_i = p_i \cdot T + p_i \cdot \bar{T}(1 - t/\bar{t}) - c_i \cdot t + c_i \cdot \bar{t}(T/\bar{T} - 1)$$

$$\text{Eq. 27} \qquad z_i = p_i \cdot \left[T + T\,(1 - t/\bar{t})\right] - c_i \cdot \left[t + \bar{t}\,(1 - T/\bar{T})\right]$$

Compare with equations 7 and 8.

$$\text{If } T_i^* = \left[T + \bar{T}(1 - t/\bar{t})\right]$$

$$\text{and } \quad t_i^* = \left[t + \bar{t}(1 - T/\bar{T})\right]$$

then we have proven that to maximize Z, the pay-off or profit to a firm, we must use:

$$\text{Eq. 28} \qquad Z_{max} = \sum_{i=1}^{m} (p_i \cdot T_i^* - c_i \cdot t_i^*)_{j=1}^{n}$$

By proof that we obtain the maximum pay-off for n-x, n-x+1, -----, n times and for m-y, m-y+1, -----, m products, as well as if $\bar{T} = \bar{t} = 1, 2,$ ----- w, and for any strategy, i.e., one and one p in each column of the matrix, only two p, only three p, etc., we have proven that Eq 11 applies to any selection or evaluation of projects, prior to starting or during development. Discounting of dollars for their present value (for profits) or future value (for costs) does not deny the proof, but supports it and makes Z_{max} more meaningful. Lastly, it should be obvious that c_{ij} and p_{ij} can now be any possible value, i.e., not necessarily equal as in the elemental matrices.

To make the model more general and exact, we introduce x_i, where x_i = o or 1 depending on whether we do not select or do select a project i, respectively. We thus have Eq 29:

$$\text{Eq. 29} \qquad Z_{max} = \sum_{i=1}^{n} x_i\,(p_i \cdot T_i^* - c_i \cdot t_i^*)_{j=1}^{n}$$

As in Figure 4 (Ref 7), for <u>a group</u> of projects, x_i = o or 1, and subject to any specific restrictions, we select G_1 by inspection out of the possible groups or combinations of all projects.

Group	Z	Total Costs
G_1	Z_1	C_1
$\vdots$	$\vdots$	$\vdots$
$\boxed{G_1}$	$\boxed{Z_1}$	$\boxed{C_1}$
$\vdots$	$\vdots$	$\vdots$
G_n	Z_n	C_n

Fig 4. Selection of group of projects.

For a large number of projects, the number of combinations would be extremely difficult to handle, therefore it is simpler to list the individual projects in a decreasing order of pay-off as determined by z (Equation 27) and successively select the project with the highest z value, next highest, etc., testing each time for compliance of the group thus being formed step-wise against any restrictions.

To illustrate the application of our proposed method, let us generate some projects randomly as shown in Table 1. The average years, $\overline{T}$ and $\overline{t}$, are equal to 3. We will also place the restriction on our problem that not more than three development projects can be handled in any one year, because of our fixed staff of three teams of engineering specialists. There are no restrictions on budget and our formula will be used as a measure of profit for one example, whereas the second example for comparison will use straight computation of profit. There is no restriction on production of number of products. Table 2 shows the computations of Z and straight profits for five projects appearing in each of five successive years. Projects not started in any year become useless in future years, but projects once started must be re-evaluated in each successive year—competing with new proposed projects. An old project is likely to be canceled by a new one if its remaining value is less than that of a new project, after a 10% margin of error in estimating is allowed. The figures have already been discounted.

Tables 3 and 4 show the computations and that our recommended method is better by 12,250 - 11, 450 = 800 thousand dollars, or $80,000/year over 10 years.

TABLE 1

Generation of Hypothetical Projects, $\bar{T} = \bar{t} = 3$

Random Numbers[1]	Project Factors p_i in \$1000	T_i in yrs.	c_i in \$1000	t_i in yrs.	Cum. F of Occurrence of $p, \bar{T}, c, t,$		Random Number Groups
	350	1	90	1	0.2		1-2
	400	2	120	2	0.4		3-4
	450	3	150	3	0.6		5-6
	500	4	180	4	0.8		7-8
	550	5	210	5	1.0		9-0
					T_i *	t_i *	Project No.
6 8 4 7	450	4	120	4	3	3	1
2 6 9 4	350	3	210	2	4	2	2
8 5 1 5	500	3	90	3	3	3	3
1 1 1 0	350	1	90	5	−1	7	No Profit
1 6 5 0	350	3	150	5	1	5	No Profit
2 6 4 5	350	3	120	3	3	3	4
9 5 2 7	550	3	90	4	2	4	5
6 7 8 9	450	4	180	5	2	4	6
9 7 3 4	550	4	120	2	5	1	7
7 3 2 0	500	2	90	5	0	6	No Profit
7 5 2 4	500	3	90	2	4	2	8
6 4 0 5	450	2	210	3	2	4	9
2 6 8 9	350	3	180	5	1	5	No Profit
4 5 4 2	400	3	120	1	5	1	10
0 1 3 9	550	1	120	5	−1	7	No Profit
8 7 3 7	500	4	120	4	3	3	11
2 0 1 1	350	5	90	1	7	−1	12
0 1 7 5	550	1	180	3	1	5	No Profit
1 9 4 7	350	5	120	4	4	2	13
3 6 1 6	400	3	90	3	3	3	14
4 5 2 4	400	3	90	2	4	2	15
4 1 9 4	400	1	210	2	2	4	No Profit
9 6 3 8	550	3	120	4	2	4	16
3 4 6 7	400	2	150	4	1	5	No Profit
2 4 8 0	350	2	180	5	0	6	No Profit
2 3 2 0	350	2	90	5	0	6	No Profit
3 8 3 1	400	4	120	1	6	0	17
6 4 0 3	450	2	210	2	3	3	18
3 6 6 9	400	3	150	5	1	5	No Profit
3 5 3 0	400	3	120	5	1	5	No Profit
6 8 6 6	450	4	150	3	4	2	19
9 0 5 5	550	5	150	3	5	1	20

TABLE 1 (Cont)

Random Numbers[1]	Project Factors P_i in \$1000	T_i in yrs.	c_i in \$1000	t_i in yrs.	T_i*	t_i*	Project No.
3580	400	3	180	5	1	5	No Profit
2210	400	2	90	5	0	6	No Profit
5072	450	5	180	1	7	-1	21
1374	350	2	180	2	3	3	22
3676	400	3	180	3	3	3	23
9182	550	1	180	1	3	3	24
5804	450	4	210	2	5	1	25

1-From pg. 176, Intro to OR, Churchman, et al.

TABLE 2

Computations of Z and Straight Profit (in $1,000 units)

Year	Project	p_iT_i*	c_it_i*	Z_i	p_iT_i*	c_it_i	Total
	1	1,350	360	990	1,800	480	1,320
	2	1,440	420	1,020	1,050	420	630
1	3	1,500	270	1,230	1,500	270	1,230
	4	1,050	360	690	1,050	360	690
	5	1,100	360	740	1,650	360	1,290
	6	900	720	180	1,800	900	900
	7	2,750	120	2,630	2,200	240	1,960
2	8	2,000	180	1,820	1,500	180	1,320
	9	900	840	60	900	630	270
	10	2,000	120	1,880	1,200	120	1,080
	11	1,500	360	1,140	2,000	480	1,520
	12	2,450	-90	2,540	1,750	90	1,660
3	13	1,400	240	1,160	1,750	480	1,270
	14	1,200	270	930	1,200	270	930
	15	1,600	180	1,420	1,200	180	1,020
	16	1,100	480	620	1,650	480	1,170
	17	2,400	0	2,400	1,600	120	1,480
4	18	1,350	630	720	900	420	480
	19	1,800	300	1,500	1,800	450	1,350
	20	2,750	150	2,600	2,750	450	2,300
	21	3,150	-180	3,330	2,250	180	2,070
	22	1,050	540	510	700	360	340
5	23	1,200	540	660	1,200	540	660
	24	1,650	540	1,110	550	180	370
	25	2,250	210	2,040	1,800	420	1,380

TABLE 3

Example Using Recommended Method

Year	Project	p_iT_i*	c_it_i*	z	Notes
1	3	1,500	270	1,230*	
	2	1,440	420	1,020	
	1	1,350	360	990	
2	3	2,000	180	1,820	
	2	1,750	210	1,540	Canceled by 7
	1	1,800	240	1,560	Canceled by 10
	7	2,750	120	2,630	
	10	2,000	120	1,880	Finished in 2nd Year
3	3	2,500	90	2,410	Finished in 3rd Year
	7	3,300	0	3,300	Finished in 3rd Year
	12	2,450	−90	2,540	Finished in 3rd Year
4	20	2,750	150	2,600	
	17	2,400	0	2,400	Finished in 4th Year
	19	1,800	300	1,500	
5	21	3,150	−180	3,330	
	20	3,300	300	3,000	
	19	2,250	150	2,100	

A. Computations and Decisions

Project	Year 1	2	3	4	5	6	7	8	9	10	
3	−90	−90	−90	+500	+500	+500					
2	−210	−210	+350	+350	+350	Cancel					
1	−120	−120	−120	−120	+450	+450	+450	+450	Cancel		
7		−120	−120	+550	+550	+550	+550				
10		−120	+400	+400	+400						
12			−90	+350	+350	+350	+350	+350			
20				−150	−150	−150	+550	+550	+550	+550	
17				−120	400	400	400	400			
19				−150	−150	−150	+450	+450	+450	+450	
21					−180	+450	+450	+450	+450	+450	
Profits			400	1,800	2,200	2,250	2,750	2,200	1,450	1,450	14,500
Costs	−420	−330	−300	−420	−480	−300					2,250
											12,250 Total

B. Charting by Year

TABLE 4

Example Using Straight Computations

Year	Project	$p_i T_i$	$c_i t_i$	Total	Notes
	1	1,800	480	1,320	
1	5	1,650	360	1,290	
	3	1,500	270	1,230	
	1	1,800	360	1,440	
	5	1,650	270	1,380	
2	3	1,500	180	1,320	Cancelled by 7
	7	2,200	240	1,960	
	1	1,800	240	1,560	
	5	1,650	180	1,470	Cancelled by 12
3	7	2,200	120	2,080	Finished in 3rd Year
	12	1,750	90	1,660	Finished in 3rd Year
	1	1,800	120	1,680	Finished in 4th Year
4	20	2,750	450	2,300	
	17	1,600	120	1,480	Finished in 4th Year
	20	2,750	300	2,450	
5	21	2,250	180	2,070	
	25	1,800	420	1,380	

A. Computations and Decisions

	Year										
Project	1	2	3	4	5	6	7	8	9	10	
1	−120	−120	−120	−120	+450	+450	+450	+450			
5	−90	−90	−90	−90	+550	+550	+550	Cancel			
3	−90	−90	−90	+500	+500	+500	Cancel				
7		−120	−120	+550	+550	+550	+550				
12			−90	+350	+350	+350	+350	+350			
20				−150	−150	−150	+550	+550	+550	+550	
17				−120	+400	+400	+400	+400			
21					−180	+450	+450	+450	+450	+450	
25					−210	−210	+450	+450	+450	+450	
Profits				900	1,750	2,300	3,200	2,650	1,450	1,450	13,700
Costs	−300	−330	−330	−390	−540	−360					2,250
											11,450 Total

B. Charting by Year

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VARIATIONS OF THE "TOLERANCE-LIMIT TEST" PROBLEM

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CONTENTS

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(140)

ABSTRACT

The following general problem is considered: A random variable x, called the attribute variable, distributed with a known frequency function and possibly contaminated with additive independent random error w, which is also distributed with known frequency function, is subjected to a test that determines whether or not $(x + w)$ lies within a specified interval; the interval may have end points which are themselves independent random variables with known frequency functions. What is the conditional frequency function of x, given that $(x + w)$ has passed the test?

An expression for the solution of the general problem is given. The problem is particularized to various special problems by imposing restrictions on the attribute random variable, the error random variable, etc. A systematic method is developed for imposing these restrictions which involves the use of Dirac delta functions. Solutions to the specialized problems are obtained utilizing specific frequency functions, and a numerical example is presented.

I. INTRODUCTION

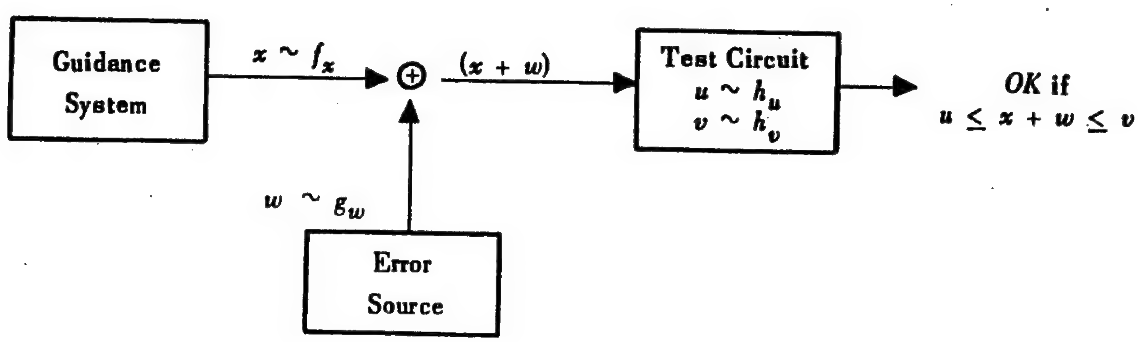
There are many engineering system test procedures which require that some attribute of the system under test be within certain tolerance limits. *Go/no go* tests of the output voltage of electronic systems fall into this category, as do many quality assurance tests used in industry. Since there are several sources of uncertainty in such test procedures (uncertainty about the exact location of the tolerance limits, uncertainty associated with the attribute being tested, etc.), it is important to make some estimate concerning just what information can be gleaned from these tests.

(1/

The problem of forming estimates may be stated probabilistically in a very general way. The purpose of this paper is to formulate and solve this general problem and to apply the solution to a number of special cases. It should be emphasized that the approach employed is based on sample statistics and not on time series analysi and the assumption which holds throughout the analysis is that the random variables discussed are time invariant

II. DISCUSSION

The discussion which follows makes use of some mathematical tools which may have become unfamiliar to the reader through disuse. For this reason a short refresher on the properties of frequency functions is include as Appendix A. In explicit terms, the question considered here is the following: a random variable x, called the attribute variable, distributed with a known *fr. f.* and possibly contaminated with additive independent random error w, also distributed with known *fr. f.*, is subjected to a test which determines whether or not $(x + w)$ lies within a specified interval; the interval may have end points which are themselves independent[1] random variables with known *fr. f.*'s. What is the conditional *fr. f.* of x, given that $(x + w)$ has passed the test; i.e., given that $(x + w)$ lies in the *distributed* test interval? For example, one preflight check of a guidance system might be the examination of a nominally zero (constant) guidance voltage with a circuit which gives an *OK to fire* indication if its input lies in a preset interval. It is assumed (1) that the voltage of interest has a known *fr. f.*, (2) that an erroı voltage (with known *fr. f.*) appears somewhere between the guidance system and test circuit and adds to the voltaį of interest, and (3) that the limits of the test interval are distributed with known *fr. f.*'s; that is, on a given test the tolerance limits are, in general, different from the nominal limits due to inaccuracies in the measuring device. Thus an *OK to fire* indication gives the information that the error-contaminated attribute variable lies between these two essentially unknown limits. This situation is shown schematically in the following sketch (it should be recalled that the voltages are assumed constant for the period under study).



[1] See footnote page 3.

(142)

We wish to determine what information an *OK* signal provides about the *fr. f.* of x; equivalently, we wish to compute the probability $p(x_1 \leq x \leq x_2 \mid u \leq x + w \leq v)$ that x lies in the specified interval (x_1, x_2), given that $(x + w)$ has passed the test. Defining $x + w = z$ and denoting events A and B as follows:

$$A: \quad x_1 \leq x \leq x_2$$

$$B: \quad u_1 \leq z \leq v$$

the desired quantity $p(x_1 \leq x \leq x_2 \mid u \leq z \leq v)$ becomes

$$p(A \mid B) = \frac{p(A \cap B)}{p(B)} \tag{1}$$

the equality being exactly the definition of conditioned probability (Ref. 1, Sec. 2.8, page 29). The *fr. f.*'s of x, w, u, and v are assumed known:

$$\begin{aligned} x &\sim f_x(\xi) \\ w &\sim g_w(\omega) \\ u &\sim h_u(\eta) \\ v &\sim h_v(\nu) \end{aligned} \tag{2}$$

Because these variables are all mutually independent,[2] their joint *fr. f.* is the product of their separate *fr. f.*'s:

$$x, w, u, v \sim f_x(\xi)\, g_w(\omega)\, h_u(\eta)\, h_v(\nu) = p(x = \xi \cap w = \omega \cap u = \eta \cap v = \nu) \tag{3}$$

What is needed to determine the quantities in Eq. (1) is not the joint *fr. f.* of x, w, u, and v but rather that of x, z, u, and v.

It is a well-known theorem of statistics (Ref. 2, Sec. 22.2, page 292) that if x and w have joint *fr. f.*

$$f_x(\xi)\, g_w(\omega)$$

the random variables x and $z = x + w$ have joint *fr. f.* (evaluated at $x = \xi$, $z = \zeta$).

[2] If u and v are not independent but have a known joint *fr. f.* $h_{u,v}(\eta, \nu)$, the factor $h_u(\eta)\, h_v(\nu)$ is simply replaced throughout this report by $h_{u,v}(\eta, \nu)$; the separate integrals with respect to η and ν in Eq. (7) et seq then combine into one double integral.

(143

$$f_x(\xi)\,g_w(\zeta-\xi)$$

It is clear that, as might be expected, x and z are not independent even though x and w were; and the presence of the factor $h_u(\eta)\,h_v(\nu)$ does not affect the argument. Thus, the joint *fr. f.* of x, z, u, and v is given by

$$x,\ z = x + w,\ u,\ v \sim f_x(\xi)\,g_w(\zeta-\xi)\,h_u(\eta)\,h_v(\nu) \tag{4}$$

Integrating over all ξ provides the joint *fr. f.* of z, u, and v (Ref. 1, Sec. 4.5, page 82)

$$z,\ u,\ v \sim \left[\int_{-\infty}^{\infty} f_x(\xi)\,g_w(\zeta-\xi)\,d\xi\right] h_u(\eta)\,h_v(\nu) \tag{5}$$

By direct application of the definition of joint *fr. f.*,

$$p(A\cap B) = p(x_1 \le x \le x_2 \cap -\infty \le z \le \infty \cap -\infty \le u \le z \cap z \le v \le \infty) \tag{6}$$

$$= \int_{\xi=x_1}^{\xi=x_2} f_x(\xi) \int_{\zeta=-\infty}^{\zeta=\infty} g_w(\zeta-\xi) \int_{\eta=-\infty}^{\eta=z} h_u(\eta) \int_{\nu=z}^{\nu=\infty} h_v(\nu)\; d\,\nu\; d\,\eta\; d\,\zeta\; d\,\xi \tag{7}$$

Similarly,

$$p(B) = p(-\infty \le z \le \infty \cap -\infty \le u \le z \cap z \le v \le \infty) \tag{8}$$

$$= \int_{\zeta=-\infty}^{\zeta=\infty} \left[\int_{\xi=-\infty}^{\xi=\infty} f_x(\xi)\,g_w(\zeta-\xi)\; d\,\xi\right] \int_{\eta=-\infty}^{\eta=z} h_u(\eta) \int_{\nu=z}^{\nu=\infty} h_v(\nu)\; d\,\nu\; d\,\eta\; d\,\zeta \tag{9}$$

Equations (7), (9) (with the order of integration altered), and (1) give

(144)

$$p(A \mid B) = p(x_1 \leq x \leq x_2 \mid u \leq z \leq v) = \frac{\int_{\xi = x_1}^{\xi = x_2} f_x(\xi) \int_{\zeta = -\infty}^{\zeta = \infty} g_w(\zeta - \xi) \int_{\eta = -\infty}^{\eta = \zeta} h_u(\eta) \int_{\nu = \zeta}^{\nu = \infty} h_v(\nu) d\nu d\eta d\zeta d\xi}{\int_{\xi = -\infty}^{\xi = \infty} f_x(\xi) \int_{\zeta = -\infty}^{\zeta = \infty} g_w(\zeta - \xi) \int_{\eta = -\infty}^{\eta = \zeta} h_u(\eta) \int_{\nu = \zeta}^{\nu = \infty} h_v(\nu) d\nu d\eta d\zeta d\xi} \tag{10}$$

Use of the extension of Eq. (A2) (Appendix A) to conditional *fr. f.*'s gives the *fr. f.* of x conditioned upon event B (B is $u \leq z = x + w \leq v$), denoted $f_x\,|_B(\xi)$:

$$f_x\,|_B(\xi) = f_x(\xi) \frac{\int_{\zeta = -\infty}^{\zeta = \infty} g_w(\zeta - \xi) \int_{\eta = -\infty}^{\eta = \zeta} h_u(\eta) \int_{\nu = \zeta}^{\nu = \infty} h_v(\nu)\, d\nu d\eta d\zeta}{\int_{\tau = -\infty}^{\tau = \infty} f_x(\tau) \int_{\zeta = -\infty}^{\zeta = \infty} g_w(\zeta - \tau) \int_{\eta = -\infty}^{\eta = \zeta} h_u(\eta) \int_{\nu = \zeta}^{\nu = \infty} h_v(\nu)\, d\nu d\eta d\zeta d\tau} \tag{10a}$$

Note that ξ is not used as a variable of integration in Eq. (10a) (τ is substituted in the denominator).

Equation (10) represents the solution of the general estimation problem posed. The problem may be particularized to various specific problems by imposing restrictions on the random variables u, v, and z and their frequency functions. For example, if u and v are allowed to be random variables, and if $z = x$, i.e., $w = 0$, the problem which is discussed in Ref. 3 is described. If u and v are required to be constants u_0 and v_0 such that $u_0 \leq x_1$ and $v_0 \geq x_2$, and if $z = x$, the classical confidence-limit problem is described (Ref. 1). A systematic approach is available for imposing these restrictions which involves the use of Dirac delta functions. It is possible to think of a constant as being a "distributed" variable whose *fr. f.* is a delta function centered at the nominal value of the constant. Thus, replacement of the appropriate *fr. f.*'s in Eq. (10) by delta functions leads in a very simple way to solution of the associated special problem. It is easily demonstrated that the δ function is indeed the frequency function of a constant. (The validity of the approach may be verified by reference to the alternate derivations of the equations which follow in Appendix B.) Thus, if α is a constant, it may also be viewed as the random variable y, where

$$y \sim \delta(y - \alpha).$$

Then the probability that y(i.e. α) lies in any interval is clearly 1 if α lies in the interval, and 0 otherwise. But

$$\int_{y_1}^{y_2} \delta(y - \alpha)\, dy = \begin{cases} 1 \text{ if } y_1 \leq \alpha \leq y_2 \\ 0 \text{ otherwise} \end{cases}$$

$$= p(y_1 \leq y \leq y_2)$$

It is also clear that

$$\int_{-\infty}^{\infty} \delta(y - \alpha)\, dy = 1$$

The general estimation problem involves additive error and distributed limits. This case reduces to that of distributed limits alone by the substitution in Eq. (2) et seq of

$$w \sim g_w(\omega) = \delta(\omega - 0) = \delta(\omega) \tag{11}$$

This substitution is equivalent to the statement that the additive error w is identically zero; z then reduces to x, and Eqs. (10) and (11) combine to yield

$$p(x_1 \leq x \leq x_2 \,|u \leq x \leq v) = \frac{\displaystyle\int_{\xi = x_1}^{\xi = x_2} f_x(\xi) \int_{\zeta = -\infty}^{\zeta = \infty} \delta(\zeta - \xi) \int_{\eta = -\infty}^{\eta = \zeta} \cdot h_u(\eta) \int_{\nu = \zeta}^{\nu = \infty} h_v(\nu)\; d\nu d\eta d\zeta d\xi}{\text{(Same as numerator with } x_1 = -\infty,\ x_2 = \infty)} \tag{12}$$

$$= \frac{\displaystyle\int_{\xi = x_1}^{\xi = x_2} f_x(\xi) \int_{\eta = \infty}^{\eta = \xi} h_u(\eta) \int_{\nu = \xi}^{\nu = \infty} h_v(\nu)\; d\nu d\eta d\xi}{\text{(Same as numerator with } x_1 = -\infty,\ x_2 = \infty)} \tag{13}$$

The *fr. f.* of x conditioned on event B (here B becomes $u \leq x \leq v$) denoted $f_x|_B(\xi)$ is given by

$$f_x|_B(\xi) = f_x(\xi) \frac{\int_{\eta=-\infty}^{\eta=\xi} h_u(\eta) \int_{\nu=\xi}^{\nu=\infty} h_v(\nu) \, d\nu d\eta}{\int_{-\infty}^{\infty} f_x(\tau) \int_{\eta=-\infty}^{\eta=\tau} h_u(\eta) \int_{\nu=\tau}^{\nu=\infty} h_v(\nu) \, d\nu d\eta d\tau} \tag{13a}$$

The note following Eq. (10a) applies.

An alternate derivation of Eq. (13), not dependent on the sneaky behavior of δ functions, is given in Appendix C.

The following substitutions give rise to the case wherein the test limits are known with certainty, but the attribute random variable x is contaminated with additive error w:

$$\begin{aligned} u &\sim h_u(\eta) = \delta(\eta - u_0) \\ v &\sim h_v(\nu) = \delta(\nu - v_0) \end{aligned} \tag{14}$$

Equation (10) becomes

$$p(x_1 \leq x \leq x_2 \,|u_0 \leq z \leq v_0) = \frac{\int_{\xi=x_1}^{\xi=x_2} f_x(\xi) \int_{\zeta=-\infty}^{\zeta=\infty} g_w(\zeta - \xi) \int_{\eta=-\infty}^{\eta=\zeta} \delta(\eta - u_0) \int_{\nu=\zeta}^{\nu=\infty} \delta(\nu - v_0) \, d\nu d\eta d\zeta d\xi}{\text{(Same as numerator with } x_1 = -\infty,\ x_2 = \infty)} \tag{15}$$

Consider the integration with respect to η in the numerator of Eq. (15). It is independent of the remainder of this expression, and its value is zero unless the argument of the δ function reaches zero – that is, unless $-\infty \leq u_0 \leq \zeta$. Similarly, the integration with respect to ν yields zero unless $\zeta \leq v_0 \leq \infty$. Thus there is no contribution to the numerator unless $u_0 \leq \zeta \leq v_0$, and the limits of the integration with respect to ζ may be confined to the interval (u_0, v_0). This argument holds for the denominator as well. Equation (16) follows immediately.

(147

$$p(x_1 \le x \le x_2 \,|\, u_0 \le z \le v_0) = \frac{\displaystyle\int_{\xi = x_1}^{\xi = x_2} f_x(\xi) \int_{\zeta = u_0}^{\zeta = v_0} g_w(\zeta - \xi)\, d\zeta d\xi}{\displaystyle\int_{\xi = -\infty}^{\xi = \infty} f_x(\xi) \int_{\zeta = u_0}^{\zeta = v_0} g_w(\zeta - \xi)\, d\zeta d\xi} \tag{16}$$

The *fr. f.* of x conditioned on event B (here B is $u_0 \le z = x + w \le v_0$), denoted $f_x\,|_B(\xi)$, becomes

$$f_x\,|_B(\xi) = f_x(\xi) \frac{\displaystyle\int_{\zeta = u_0}^{\zeta = v_0} g_w(\zeta - \xi)\, d\zeta}{\displaystyle\int_{\tau = -\infty}^{\tau = \infty} f_x(\tau) \int_{\zeta = u_0}^{\zeta = v_0} g_w(\zeta - \tau)\, d\zeta d\tau} \tag{16a}$$

The note following Eq. (10a) applies.

Equation (10) reduces to the simplest case, that of the classical confidence limit problem, upon the substitutions:

$$\left.\begin{aligned} w &\sim g_w(\omega) = \delta(\omega) \\ u &\sim h_u(\eta) = \delta(\eta - u_0) \\ v &\sim h_v(\nu) = \delta(\nu - v_0) \end{aligned}\right\} \tag{17}$$

and the specification that x_1 and x_2 be within (u_0, v_0). Equation (10) becomes:

$$p(x_1 \le x \le x_2 \,|\, u_0 \le x \le v_0) = \frac{\displaystyle\int_{\xi = x_1}^{\xi = x_2} f_x(\xi) \int_{\zeta = -\infty}^{\zeta = \infty} \delta(\zeta - \xi) \int_{\eta = -\infty}^{\eta = \zeta} \delta(\eta - u_0) \int_{\nu = \zeta}^{\nu = \infty} \delta(\nu - v_0)\, d\nu d\eta d\zeta d\xi}{\text{(Same as numerator with } x_1 = -\infty,\ x_2 = \infty)}$$

(18)

(148)

The reduction of the numerator is as follows:

$$\int_{\xi=x_1}^{\xi=x_2} f_x(\xi)\int_{\zeta=-\infty}^{\zeta=\infty} \delta(\zeta-\xi)\int_{\eta=-\infty}^{\eta=\zeta} \delta(\eta-u_0)\int_{\nu=\zeta}^{\nu=\infty} \delta(\nu-v_0)\,d\nu\, d\eta\, d\zeta\, d\xi = \int_{\xi=x_1}^{\xi=x_2} f_x(\xi)\int_{\zeta=u_0}^{\zeta=v_0} \delta(\zeta-\xi)\,d\zeta\, d\xi \tag{19}$$

as in Eq. (16), but the integral

$$\int_{\zeta=u_0}^{\zeta=v_0} \delta(\zeta-\xi)\, d\zeta = 1 \tag{20}$$

since $u_0 \leq \xi \leq v_0$.

The numerator becomes

$$\int_{\xi=x_1}^{\xi=x_2} f_x(\xi)\, d\xi \tag{21}$$

The denominator is reduced similarly.

$$\int_{\xi=-\infty}^{\xi=\infty} f_x(\xi)\int_{\zeta=-\infty}^{\zeta=\infty} \delta(\zeta-\xi)\int_{\eta=-\infty}^{\eta=\zeta} \delta(\eta-v_0)\int_{\nu=\zeta}^{\nu=\infty} \delta(\nu-v_0)\,d\nu\, d\eta\, d\zeta\, d\xi = \int_{\xi=-\infty}^{\xi=\infty} f_x(\xi)\int_{\zeta=u_0}^{\zeta=v_0} \delta(\zeta-\xi)\, d\zeta\, d\xi \tag{22}$$

The integral on the right side of Eq. (22) has values only in the interval (u_0, v_0) by reason of the definition of the delta function. It may be replaced by the equivalent relation

(1/

$$\int_{\xi = u_0}^{\xi = v_0} f_x(\xi)\, d\xi \tag{23}$$

Equation (18) becomes

$$p(x_1 \leq x \leq x_2 \,|\, u_0 \leq x \leq v_0) = \frac{\displaystyle\int_{\xi = x_1}^{\xi = x_2} f_x(\xi)\, d\xi}{\displaystyle\int_{\xi = u_0}^{\xi = v_0} f_x(\xi)\, d\xi} \tag{24}$$

The *fr. f.* of x conditioned on event B (where B is now $u_0 \leq x \leq v_0$), denoted $f_{x|B}(\xi)$, is

$$f_{x|B}(\xi) = \frac{f_x(\xi)}{\displaystyle\int_{\tau = u_0}^{\tau = v_0} f_x(\tau)\, d\tau} \tag{24a}$$

The note following Eq. (10a) again applies.

It is perhaps instructive to make the preceding discussion more concrete by assigning specific functional forms to the *fr. f.*'s and applying them to the example outlined on page 2 of this report. Suppose the nominally zero guidance voltage has a rectangular distribution about zero with end points at ±1 volt. Thus

$$x \sim f_x(\xi) = \begin{cases} \dfrac{1}{2} & -1 \leq \xi \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

Let the error voltage be normally distributed about zero with standard deviation σ_w.

(150)

$$w \sim g_w(\omega) = \phi(\omega;\ 0,\ \sigma_w) = \frac{1}{\sqrt{2\pi\sigma_w}} \exp^{-\frac{\omega^2}{2\sigma_w^2}} \tag{26}$$

Let the limits of the test interval be normally distributed about the nominal limits u_0 and v_0 with standard deviations σ_u and σ_v.

$$\begin{aligned} u &\sim \phi(\eta;\ u_0,\ \sigma_u) = h_u(\eta) \\ v &\sim \phi(\nu;\ v_0,\ \sigma_v) = h_v(\nu) \end{aligned} \tag{27}$$

Equations (26) and (27) combine with Eq. (10) to give

$$p(x_1 \le x \le x_2 | u \le z \le v) = \frac{\int_{\xi = x_1}^{\xi = x_2} f_x(\xi) \int_{\zeta=-\infty}^{\zeta=\infty} \phi(\zeta - \xi;\ 0,\ \sigma_w) \int_{\eta=-\infty}^{\eta=\zeta} \phi(\eta;\ u_0 \sigma_u) \int_{\nu=\zeta}^{\nu=\infty} \phi(\nu;\ v_0, \sigma_v) d\nu d\eta d\zeta d\xi}{\text{(Same as numerator with } x_1 = -\infty,\ x_2 = \infty)} \tag{28}$$

It is required that $-1 \le x_1 \le x_2 \le +1$ (the problem is soluble without this restriction, but attention here will be confined to this case); then $f_x(\xi)$ is simply ½ in the expression in the numerator; the integrand in the denominator is zero unless $-1 \le \xi \le +1$, so that the interval of integration with respect to ξ may be taken to be $(-1, 1)$. Over this interval $f_x(\xi)$ is ½, cancelling the ½ in the numerator and giving

$$p(x_1 \le x \le x_2 | u \le z \le v) = \frac{\int_{\xi = x_1}^{\xi = x_2} \int_{\zeta=-\infty}^{\zeta=\infty} \phi(\zeta - \xi;\ 0,\ \sigma_w) \int_{\eta=-\infty}^{\eta=\zeta} \phi(\eta;\ u_0,\ \sigma_u) \int_{\nu=\zeta}^{\nu=\infty} \phi(\nu;\ v_0 \sigma_v)\, d\nu d\eta d\zeta d\xi}{\text{(Same as numerator with } x_1 = -1,\ x_2 = 1)} \tag{29}$$

(151

The order of integration is changed, and some facts regarding ϕ and Φ (defined in the Nomenclature) are used to give

$$p(x_1 \leq x \leq x_2 \,|\, u \leq z \leq v) = \frac{\displaystyle\int_{\zeta=-\infty}^{\zeta=\infty}\int_{\xi=x_2}^{\xi=x_2} \phi(\xi;\ \zeta,\ \sigma_w)\ \Phi\ (\zeta;\ u_0,\ \sigma_u)\left[1-\Phi\ (\zeta;\ v_0,\ \sigma_v)\right]\ d\xi d\zeta}{\text{(Same as numerator with } x_1 = -1,\ x_2 = 1)} \tag{30}$$

or

$$p(x_1 \leq x \leq x_2 \,|\, u \leq z \leq v) = \frac{\displaystyle\int_{\zeta=-\infty}^{\zeta=\infty}\left[\Phi(x_2;\ \zeta,\ \sigma_w) - \Phi(x_1;\ \zeta,\ \sigma_w)\ \Phi(\zeta;\ u_0,\ \sigma_u)\right]\left[1-\Phi(\zeta;\ v_0\ \sigma_v)\right]\ d\zeta}{\text{(Same as numerator with } x_1 = -1,\ x_2 = 1)} \tag{31}$$

Equation (31), while somewhat lengthy, is amenable to numerical integration, converging rapidly to the correct value even for relatively large $\Delta\zeta$ intervals. The function Φ is, of course, tabulated (Ref. 1, Table II, page 423). The case of normally distributed test limits with an error-free, normally distributed attribute variable is exhaustively treated in Ref. 3.

Another example which is of interest is the fixed certain limit problem with a triangularly distributed attribute variable and normally distributed error. T. W. Hamilton has treated this problem in a recent practical application. It will be repeated here using the techniques developed herein. Let

$$x \sim f_x(\xi) = \begin{cases} \xi + 1, & -1 \leq \xi \leq 0 \\ -\xi + 1, & 0 \leq \xi \leq 1 \\ 0 \text{ otherwise} & \end{cases} \tag{32}$$

$$w \sim g_w(\omega) = \phi(\omega;\ 0,\ \sigma_w) \tag{33}$$

Equation (33) may be inserted in Eq. (16) to give

(152)

$$p(x_1 \le x \le x_2 \mid u_0 \le z \le v_0) = \frac{\int_{\xi = x_1}^{\xi = x_2} f_x(\xi) \int_{\zeta = u_0}^{\zeta = v_0} \phi(\zeta - \xi;\ 0,\ \sigma_w)\ d\zeta d\xi}{\int_{\xi = -1}^{\xi = 1} f_x(\xi) \int_{\zeta = u_0}^{\zeta = v_0} \phi(\zeta - \xi;\ 0,\ \sigma_w)\ d\zeta d\xi} \tag{34}$$

Note that the ξ integration interval in the denominator has been taken to be $(-1, 1)$, since $f_x(\xi) = 0$ outside this interval. The formulas involving ϕ and Φ (see Nomenclature) are employed to give

$$p(x_1 \le x \le x_2 \mid u_0 \le z \le v_0) = \frac{\int_{\xi = x_1}^{\xi = x_2} f_x(\xi) \int_{\zeta = u_0}^{\zeta = v_0} \phi(\zeta;\ \xi,\ \sigma_w)\ d\zeta d\xi}{\int_{\xi = -1}^{\xi = 1} f_x(\xi) \int_{\zeta = u_0}^{\zeta = v_0} \phi(\zeta;\ \xi,\ \sigma_w)\ d\zeta d\xi} \tag{35}$$

$$= \frac{\int_{\xi = x_1}^{\xi = x_2} f_x(\xi) \left[\Phi(v_0;\ \xi,\ \sigma_w) - \Phi(u_0;\ \xi,\ \sigma_w)\right] d\xi}{\int_{\xi = -1}^{\xi = 1} f_x(\xi) \left[\Phi(v_0;\ \xi,\ \sigma_w) - \Phi(u_0;\ \xi,\ \sigma_w)\right] d\xi} \tag{36}$$

Equation (36), with $f_x(\xi)$ explicitly given by Eq. (32), may be integrated numerically in specific applications. For example, let

$$u_0 = -0.5 = x_1$$

$$v_0 = +0.5 = x_2$$

$$\sigma_w = 0.1$$

Trapezoidal integration (with $\Delta\xi = 0.25$) yields

$$p(-0.5 \le x \le +0.5 \mid -0.5 \le z \le +0.5) = 0.92$$

(The error due to the use of trapezoidal integration will be no larger than 0.005.)

The interpretation of this result is that one may be "92% confident" that the attribute variable lies within the tolerance limits if the error standard deviation is as stated, given that the test has been passed.

NOMENCLATURE

$x,\ y,\ w,\ z,\ u,\ v$	random variables
$\xi,\ \lambda,\ \omega,\ \zeta,\ \eta,\ \nu$	the corresponding running variables or variables of integration
$A,\ B,\ C,\ \cdots$	events
$\frown$	"and" = "intersection" (of two events)
$p(A)$	probability of event A occurring
$p(A \frown B)$	probability of both A and B occurring
$p(A \mid B)$	conditional probability of A on B = probability of A occurring given that B has occurred
fr. f.	frequency function
$\sim$	distributed with *fr. f.*
$f_x(\xi),\ g_w(\omega),\ \cdots$	*fr. f.* of $x,\ w,\ \cdots$ evaluated at $\xi,\ \omega,\ \cdots$; i.e., $f_x(\xi) = p(x = \xi)$ or $p(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} f_x(\xi)\, d\xi$, etc.
$f_{x\mid E}(\xi)$	conditional *fr. f.* of x on event E evaluated at ξ; i.e., $f_{x\mid E}(\xi) = p(x = \xi \mid E)$, etc.
$\phi(\xi;\ \mu,\ \sigma)$	normal *fr. f.* with mean μ and standard deviation σ, evaluated at ξ $= \frac{1}{\sqrt{2\pi\sigma}} \exp -\frac{1}{2}\left(\frac{\xi - \mu}{\sigma}\right)^2$. Note that $\phi(\xi;\ \mu,\ \sigma) = \phi(\mu;\ \xi,\ \sigma) = \phi(\xi - \mu;\ 0,\ \sigma) = \phi(\mu - \xi;\ 0,\ \sigma)$
$\Phi(\xi;\ \mu,\ \sigma)$	cumulative normal distribution function with mean μ and standard deviation σ, evaluated at $= \int_{-\infty}^{\xi} \phi(\tau;\ \mu,\ \sigma)\, d\tau = 1 - \int_{\xi}^{\infty} \phi(\tau;\ \mu,\ \sigma)\, d\tau$; $\int_{\xi_1}^{\xi_2} \phi(\xi;\ \mu,\ \sigma)\, d\xi = \Phi(\xi_2;\ \mu,\ \sigma) - \Phi(\xi_1;\ \mu,\ \sigma)$
$(x_1,\ x_2)$	signifies the closed interval lying between x_1 and x_2; i.e., x in $(x_1,\ x_2)$ signifies $x_1 \leq x \leq x_2$

(151)

APPENDIX A

A Short Refresher on the Properties of Frequency Functions

A random or distributed variable is one which may take on any value within a specified (finite or infinite) range. To any interval in this range there corresponds a probability of the random variable falling within the interval. The *fr. f.* f_x of a continuous random variable x is a function whose integral over any interval (x_1, x_2) equals the probability that x lies in (x_1, x_2):

$$p(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} f_x(\tau)\, d\tau \tag{A1}$$

By differentiating this expression with respect to x_2 and setting $x_2 = \xi$, $x_1 = -\infty$ (which is traditional),

$$f_x(\xi) = \frac{d}{d\xi}\left[p(-\infty \leq x \leq \xi)\right] \tag{A2}$$

Since probabilities are always non-negative, and since the probability of any event which is certain to occur (such as $-\infty \leq x \leq \infty$) is 1, any *fr. f.* $f_x(\xi)$ satisfies

$$\left.\begin{aligned} f_x(\xi) &\leq 0 \\ \int_{-\infty}^{\infty} f_x(\xi)\, d\xi &= 1 \end{aligned}\right\} \tag{A3}$$

Frequency functions of several variables are defined as follows: the joint *fr. f.* $f_{x, y, \cdots}(\xi, \lambda, \cdots)$, integrated over (x_1, x_2), (y_1, y_2), $\cdots$, gives the probability that x lies in (x_1, x_2) *and* y lies in (y_1, y_2) *and* $\cdots$:

$$p(x_1 \leq x \leq x_2 \cap y_1 \leq y \leq y_2 \cap \cdots) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} \cdots f_{x, y, \cdots}(\xi, \lambda, \cdots)\, d\xi\, d\lambda \cdots \tag{A4}$$

If only certain of the variables in the expressions above -- say x and z -- are of interest, their joint *fr. f.* $f_{x, z}(\xi, \zeta)$ is obtained by integrating $f_{x, y, z, \cdots}(\xi, \lambda, \zeta, \cdots)$ over the entire range ($-\infty$ to $+\infty$) of all the remaining variables:

(155)

$$f_{x,z}(\xi, \zeta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots f_{x,y,z,\cdots}(\xi, \lambda, \zeta, \cdots)\, d\lambda \cdots\cdot \tag{A5}$$

and

$$p(x_1 \leq x \leq x_2 \cap z_1 \leq z \leq z_2) = \int_{x_1}^{x_2} \int_{z_1}^{z_2} f_{x,z}(\xi, \zeta)\, d\xi d\zeta = \int_{x_1}^{x_2} \int_{-\infty}^{\infty} \int_{z_1}^{z_2} f_{x,y,z,\cdots}(\xi, \lambda, \zeta, \cdots)\, d\xi d\lambda d\zeta \cdots\cdot \tag{A6}$$

(A complete discussion of frequency functions is given in Chapters 3 and 4 of Ref. 1.)

APPENDIX B

The Additive Error and Fixed Limits Problem

The following derivation covers the case wherein the variable of interest plus independent additive random error is subjected to a test with fixed limits (Eqs. 14, 15, and 16). The assumptions are

$$x \sim f_x(\xi), \qquad w \sim g_w(\omega)$$

and the answer sought is

$$p(x_1 \leq x \leq x_2 | u_0 \leq x + w \leq v_0) = p(A|B) = \frac{p(A \cap B)}{p(B)} \tag{B1}$$

as in Appendix C.

The quantities x and w are independent, so that

$$x, w \sim f_x(\xi)\, g_w(\omega) \tag{B2}$$

Defining

$$z = x + w \tag{B3}$$

and using the formulas of Ref. 2(Sec. 22.2, page 292) provides

$$x, z = x + w \sim f_x(\xi)\; g_w(\xi - \zeta) \tag{B4}$$

Integrating x out provides the *fr. f.* for z:

$$z = x + w \sim \int_{-\infty}^{\infty} f_x(\xi)\; g_w(\xi - \zeta)\, d\xi \tag{B5}$$

Then

$$p(x_1 \leq x \leq x_2 | u_0 \leq z \leq v_0) = \frac{p(x_1 \leq x \leq x_2 \cap u_0 \leq z \leq v_0)}{p(u_0 \leq z \leq v_0)} \tag{B6}$$

or

$$p(x_1 \leq x \leq x_2 | u_0 \leq z \leq v_0) = \frac{\int_{\xi = x_1}^{\xi = x_2} f_x(\xi) \int_{\zeta = u_0}^{\zeta = v_0} g_w(\xi - \zeta)\; d\zeta d\xi}{\int_{\xi = -\infty}^{\xi = \infty} f_x(\xi) \int_{\zeta = u_0}^{\zeta = v_0} g_w(\xi - \zeta)\; d\zeta d\xi} \tag{B7}$$

as before (note that the order of integration has been reversed in the denominator).

APPENDIX C

The Distributed Limits Problem

The following is an alternate derivation of the case wherein the variable of interest, uncontaminated by any additive error, is subjected to a test with distributed limits. This problem is worked out (using δ functions) in Eqs. (11), (12), and (13). What is given is

$$x \sim f_x(\xi), \qquad u \sim h_u(\eta), \qquad v \sim h_v(\nu)$$

and the answer required is

$$p(x_1 \leq x \leq x_2 | u \leq x \leq v) = p(A|B) = \frac{p(A \cap B)}{p(B)} \tag{C1}$$

defining A as $x_1 \leq x \leq x_2$, B as $u \leq x \leq v$, and using Eq. (1). Since x, u, and v are assumed mutually independent,

$$x, u, v \sim f_x(\xi)\ h_u(\eta)\ h_v(\nu) \tag{C2}$$

and

$$p(x_1 \leq x \leq x_2 | u \leq x \leq v) = \frac{p(x_1 \leq x \leq x_2 \cap -\infty \leq u \leq x \cap x \leq v \leq \infty)}{p(-\infty \leq x \leq \infty \cap -\infty \leq u \leq x \cap x \leq v \leq \infty)} \tag{C3}$$

or

$$p(x_1 \leq x \leq x_2 | u \leq x \leq v) = \frac{\displaystyle\int_{\xi = x_1}^{\xi = x_2} f_x(\xi) \int_{\eta = -\infty}^{\eta = \xi} h_u(\eta) \int_{\nu = \xi}^{\nu = \infty} h_v(\nu)\ d\nu d\eta d\xi}{(\text{Same as numerator with } x_1 = -\infty,\ x_2 = \infty)} \tag{C4}$$

as before.

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ANALYSIS OF DEMAND RATES

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1. This report presents an analysis of demand data and demand variability of Ordnance secondary items used in organizational and field maintenance.

2. Out of approximately 400,000 Ordnance items that appear in supply catalogs, about 50,000 are considered active and authorized for stockage at field level. These are the items that are available at direct support companies and post ordnance installations and meet the criterion of having been demanded by some using unit a minimum of 3 times in 180 days. This criterion of activity is currently under revision so as to include other frequencies of demands for a 360 day period. If an item does not meet the criterion of activity it is considered a "fringe" item and only stocked by the depots.

3. The degree of activity as measured by the quantitative demand on about 54,000 AFSCS active items, as indicated in Stockage List #8 for Calendar Year 1958 prepared by the Ordnance Supply Analysis Agency, is approximately as follows:

Items with zero annual demand	50%
Items with a frequency of demand less than 6 times in 360 days	25%
Items with more than 6 demands per year	25%

The relatively high percentage of items with either zero demand or six or less number of demands per year may be due to the following reasons:

a. Items that are considered "stand by" or insurance type items and included in AFSCS list even if they do not meet the criterion of activity.

b. Items that no longer meet the criterion of activity but apply to end items still in the field and are being used to attrition.

c. Items that pertain to newly introduced major items.

d. Items that the NICP consider superseded-by or exhausted-to some preferred item and are deleted from the AFSCS. The stockage list will mention the preferred item which may have very little demand during the period.

e. Items added to stockage list after OSAA cut-off date for processing demands (about three month interval).

f. Items that reporting station may neglect to omit after it has become inactive.

It can be said that out of 54,000 active items only about 25% or 13,500 of them will have some degree of activity. A good 10% of these items will probably have small quantitative demand which, when related to the density of the major items, may give a rate very close to zero. It thus appears that rates will be computed on about 12,000 items, that is, on about 22% of the active items appearing in the Army Field Stock Control System.

4. The analysis to be described in this research will refer to the 22% of items that show some degree of activity during the year.

5. The data needed for the computation of demand rates originates from the Army Field Stock Control System of recording demands. Three basic decks of cards are needed for rate computations.

a. Demand data sent by direct support organizations from 7th Army and by Post Ordnance installations from CONUS and other theatres.

b. Density of major item supported by the installtions. This density report also includes a tentative classification of the type of activity to which the major item is subjected.

c. Application deck sent by the National Inventory Control Points.

d. There are other decks of cards which are used in the EAM computational process. These decks are needed for updating the numbers that identify the secondary items, and for the determination of items that become obsolete, are lost to another technical service, or are superseded by other items.

6. The demand data analyzed in this research is the demand at Post Ordnance installations. As such, it reflects very closely the actual use of the item at field level. An original demand as shown in a "single line requisition" (Form 1546) is summarized monthly at either the post ordnance installation or the direct support organization in an EAM card and sent to OSAA for analysis. Six monthly demands are available in each demand data card. When these cards are shipped to OSAA, an accompanying letter called the "cover letter", showing the volume of major items supported during that period, is also sent to OSAA for review.

7. Approximately 114 units, world wide, will send above demand data information for analysis. The breakdown of these units is as follows: 45 from CONUS, 37 from 7th Army, 21 from the rest of Europe, 7 from AFFE and 4 from Alaska and other areas. Approximately 720,000 EAM cards with demand information are received every year. This figure includes 300,000 from CONUS, 170,000 from 7th Army and 250,000 from the rest of the world. About 40,000 density cards are also being received from the using organizations. The volume of cards of the other decks needed for rate computation is as follows: application deck - 200,000, "change-to" file - 170,000, "transfer and change-note" file - 370,000.

8. For classification purposes, the factors affecting demand variability can be divided into two general categories:

a. Major factors for which a weight can be determined by means of statistical methods on the basis of data gathered from the using units.

b. Other factors which will be categorized as possible causes for unexplained fluctuations (random errors) in the reported data. Analysis of these factors will facilitate the elimination of spurious information due to reporting errors and also provide a basis for studying the causes of improper demands within a particular area or unit.

9. Major factors affecting the demand rate. Three major factors are scheduled to be considered in the rate computation. These are: density of major items supported, type of utilization of the major item and climatic conditions.

a. For some type of major items it is possible that either the number of rounds fired (weapons) or miles traveled (wheeled or tracked equipment) or hours of operation should be preferred to the density supported by the reporting units. No data of these factors is available on a regular basis from all the reporting units. Collection of this type of data is a formidable task and it is doubtful that reliable figures will be obtained from the using units. The density of a major item is a broad index of consumption; it is readily available and can be easily used for prediction of future demands.

b. The type of utilization of the equipment can be measured from information that may be available on maneuvers, field exercises and training programs. Other types of utilization are: internal garrison duties and stand-by status.

c. Knowledge of the geographical area and season in which the equipment was used will determine several categories of climatic conditions. It is possible that, on final analysis, only extreme climatic conditions will affect significantly the demand rate computed.

10. Analysis of factors causing random errors.

a. Factors affecting demand variability. Some of these factors will tend to indicate a demand from the field larger than the actual use of the item; as, for instance, artificial demands due to out-of-stock position, cancellations, artificial demands due to availability of funds, unauthorized applications. Other factors may tend to show a demand smaller than the actual consumption, as, for instance, a demand filled by a unit pack instead of its original number, causing a lack of report of future demands, fringe demands from stations where item is not on their stockage list, artificial disposition of stock. Other factors may tend to work in either direction; for instance, incorrect classification of recurrent vs. nonrecurrent demands.

However, due to the fact that these demands* are analyzed at post ordnance level, the effect of above factors is not considered to be as important as in the case when they are analyzed at depot level. The only exception may be the "locally fringe" demands which are presently not used in the computation of demand rates due to processing difficulties. An item is considered locally fringe when it is stocked only at some of the reporting stations. Demand rates based on "active" demands only, should in general be lower than the demand rate obtained when both active and fringe demands are included in the calculations. These fringe demands are annotated on the forms 1546 which are also sent to OSAA for analysis. A future report will show the effect of these demands on the demand rate.

b. Factors affecting density variability. The density submitted by the using units should be an average covering the period of the report. Density information is available on a semi-annual basis and it is suspected that the figure quoted is the amount supported as to the date of report, which may not cover any density fluctuation during that period. It is also known that some units evacuate equipment to a different echelon of maintenance and continue to report these major items as supported equipment. Errors of this type may be eliminated if the control of major items is part of an EAM card with correct identification prepunched, and it is required that the reporting units post quarterly figures in density supported and information of evacuated equipment and overflow maintenance.

c. Other factors causing random errors.

(1) Quality of material, operations and maintenance. This classification includes those factors related to the quality of the repair part on items under consideration, to the quality of the operator of the equipment and to the existence of and the adherence to a prescribed maintenance plan covering the equipment. Specific factors are:

Quality of items: a. Original specifications or design not adequate for application.

b. Use of sub-standard material obtained through local purchase or manufacture.

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* The effect of these factors should be investigated by conducting a controlled experiment at field level. A team of research workers should conduct a survey at some randomly selected reporting stations throughout the world and examine the behavior of the demands on a random sample of secondary items.

Quality of operators:	a.	Abuse of equipment
	b.	Degree of skill or training involved
Quality of maintenance:	a.	Facilities
	b.	Skill

(2) Reporting errors. This broad category contains those factors caused by human error in gathering and recording data relative to demand and densities. At best, a well-designed reporting system can be expected only to minimize errors in this respect. The amount of control necessary to eliminate errors entirely would be, in general, prohibitive in manpower and costs. When errors of this type are extreme and obvious it may be necessary to withdraw the spurious data from the computation.

11. As mentioned above, other decks of cards are needed for the rate computation. Mention should be made of the condition of the application deck. Application data received from the National Inventory Control Points is still incomplete and with many errors. The original Master Application Deck consists of approximately 800,000 cards. By eliminating the major item applications that are not used in the field, and by grouping those major items that are similar in nature, the Technical Section of OSAA was able to reduce the application deck to about 200,000 cards.

12. Age of Equipment. A demand rate is obtained based on demand and density reported under different conditions of age and usage of the equipment fleet. It is possible that an isolated unit may have either all new or old equipment during a year, but it is more likely that both types of equipment were represented in the average density of the majority of the reporting units.

The assumption is made that the average age of the equipment from all the reporting units will remain approximately the same from year to year. However, in the analysis of the demand rates it will be possible to single out those units with substantially low or high rates when these rates are due to new and old equipment, respectively. Substantial changes of the average age of the fleet will also be reflected in the change of the demand rate which will be computed every six or twelve months.

13. Demand and density data for approximately three years of data will be available for review by October 1959. This report will give a partial analysis of the results obtained in a sample of 75 items selected by the Equipment Specialists of the Technical Section of OSAA. This sample was selected on the basis of the activity of the item and should not be considered as a random sample from the population of active items.

The percentage distribution of the 75 items according to unit price and volume of demand is as follows:

UNIT PRICE $	DEMANDS 1-99	100-999	1.000-9.999	10,000-99.999	TOTAL
.000 - .010	0 %	0 %	0 %	0 %	0 %
.011 - .100	11	2.3	4.6	0	8
.101 - 1.000	4.6	9.2	12.6	1.2	27.6
1.001 - 10.000	6.9	20.7	9.2	1.1	37.9
10.001 - 100.000	4.6	10.3	2.3	12	18.4
100.001 - 1000.00	1.1	1.2	3.5	1.1	6.9
1000.001 - 10,000.00	1.2	0	0	0	1.2
TOTAL	19.5	43.7	32.2	5	100.00

The composition of the sample selected from each group of major activity is as follows:

6	items from Small Arms
12	items from Antiaircraft Artillery
10	items from Combat Vehicles
9	items from Non-Combat Vehicles
9	items from Electronic Fire Control
12	items from Optical Fire Control
10	items from Guided Missiles
7	items from Hardware
TOTAL: 75	

An additional random sample of 50 items will be incorporated into this study, results of which will be shown in a future report. This sample was selected by M.I.T. for the research in secondary item supply control for the Inventory Management Project, Office of Ordnance Research.

14. Three years of field demand-density data, covering FY '56, FY '57 and FY '58 was furnished RCA for the purpose of computing demand rates. The RCA rate contract will be completed in September 1959. The demand rates will be listed in two sequences: the secondary item stock number sequence followed by its applicable major items and the major item stock number sequence followed by all of its applicable secondary items. The proposed rate format is as follows:

a)

SISN	NOUN	U/M	SNL	Sg	QSq	MISN	QPMI
2540 770 2985	CAP ASSY	EA	G501	306	1	2320 223 8120	1
			G254	3061	2	2320 796 5986	2

CONUS FY'56		CONUS FY'57		7TH ARMY FY'57		CONUS FY'58		7TH ARMY FY'58	
RATE	ST.ERROR	RATE	ST.ERROR	RATE	ST.ERROR	RATE	ST.ERROR	RATE	ST.ERROR
.31	.04	.38	.08	.35	.07	.38	.08	.41	.09
.62	.09	.74	.11	.71	.10	.74	.11	.75	.11

b)

MISN	NOMENCLATURE	SNL	SISN	NOUN	U/M	QPMI
1015 591 0186	MORTAR 4PT 2IN M2 W-E	A070	1015 713 4391	KNOB	EA	1

CONUS FY'56		CONUS FY'57		7TH ARMY FY57		CONUS FY'58		7TH ARMY FY'58	
RATE	ST.ERROR	RATE	ST.ERROR	RATE	ST.ERROR	RATE	ST.ERROR	RATE	ST.ERROR
.50	.09	.58	.11	.45	.09	.65	.13	.51	.10

15. Computation of the Demand Rate for a single application item. Exhibit No. 1 shows the data, formulas and charts associated with the computation of a single application rate. It includes:

a. A list of the post ordnance units in CONUS that reported demand-density data for one year. The list contains the following information: Column #1 is the unit designators, Column #2 is the actual reported yearly densities, Column #3 is the scaled densities and Column #4 is the yearly demands.

b. The chart contains the following information:

(1) SISN
(2) Noun
(3) Major Item Application
(4) Annual demands - vertical scale
(5) Annual densities - horizontal scale
(6) Rate Line - line from origin
(7) Demand-Density Points for each CONUS unit
(8) Confidence Belt - Broken line which depicts an area of two standard errors above and below the rate line.
(9) Basic Demand Rate = b
(10) Demand Rate for 100 Major Items for 15 days = DR
(11) Standard Error = s
(12) Standard Error of Estimate = s_b
(13) Standard Error of Estimate for 100 Major Items for 15 days s_{DR}.
(14) Coefficient of variation = c

The demand-density chart furnishes a graphic portrayal of demand-density data for post ordnance units in CONUS for fiscal year 1957 as received by Raritan Arsenal.

Demand-density points falling within the confidence belt are considered to be acceptable points. Those points falling beyond the confidence belt are considered unusual or abnormal points in that the rate of demand is above or below normal in respect to the number of major items supported. The unusual points in the chart such as unit designators 09038 and 05010 should be investigated to determine the reasons for the abnormalities e.g. type of activity, maintenance policies, climatic conditions, etc.

c. Distribution of Residuals. This chart shows a frequency distribution of residuals in terms of the standard error. If the assumption of normality is satisfied, the residuals should be symmetrically distributed about zero with 99% of the residuals falling between -3 and 3. For this item, the assumption of normality is not unreasonable. (See Exhibit 1)

16. Computation of a demand rate for a multiple application item. (Exhibit 2) A secondary item having two or more applications is called a multiple application. Exhibit #2 gives an example of a multiple application item. The secondary item, Fuse, Cartridge Lamp 25V applies to the following two major items: F342 which has a quantity per major item of one and Y004 which has a quantity per major item of two. The densities of the two applications are added, after weighing each application according to the quantity per major item, and the rate is computed following the techniques of a single application item. Thus, U_1 and U_2 represent the densities of the major item applications, F342 and Y004 respective, and m_1 and m_2 represent the quantity per major items (number of secondary items incorporated into the major items). The combined density is computed according to the following method:

$$U = m_1 U_1 + m_2 U_2$$

This method assumes that if a secondary item is used in two or more different types of major items, the rate of use will be linearly related to the quantity of secondary item incorporated in each of the major items. Variations due to different quantities (m) in which the secondary item applies to each major item are properly accounted by weighing each density by the proper value of m before obtaining the combined density (U). Thus, the Fuse, Cartridge Lamp 25V applies one (1) time in an F342 and two (2) times in a Y004. Unit designator 04142 (Exhibit #2) reports a yearly demand of 144 fuses and a density of five (5) F342 and twelve (12) Y004, then the assumption is that the 144 fuses were used in a combined density of (1) (5) + 2(12) = 29. That is, the 144 fuses were used in 29 different locations.

The demand rate obtained after the densities are combined represents the demands for 100 locations for 15 days. To obtain the demand rate for 100 major items, the demand rate per location is multiplied by the quantity per major item.

17. Using the field demands for FY 1957, the frequency distribution of number of applications is as follows:

Number of applications:	
1	32.1%
2	19.6%
3	10.3%
4	3.4%
5	4.0%
6	5.0%
7	6.0%
8	1.6%
9	1.6%
10	5.6%
more than 10	10.8%

18. Analysis of Rates. A complete analysis of the demand rates calculated for FY 1956, 1957 and 1958 will be shown in Statistical Progress Report No. 8 to be published in the near future. We should now briefly summarize the results obtained using the sample of 75 items from demand-density data from CONUS for FY 1957.

a. Coefficient of Variance

(1) The coefficient of variance (c) is defined as the ratio of the standard of estimate error to the demand rate. The distribution is as follows:

c	Related Frequency
0-4	0%
5-9	16%
10-14	28%
15-19	11%
20-24	11%
25-29	9%
30-34	7%
35-39	2%
40-44	2%
45-80	14%

(2) It is seen that 44% of the items have a c less than 15%, 31% have a c between 15% and 30% and 25% have a c of more than 30%.

(3) Items with a coefficient of variance of less than 15% are considered statistically reliable. Figures 1 through 8 show examples of those items. If points with residuals of ± 2 standard errors (points outside the confidence belt) are eliminated from the calculation, 54% of the items in this group will have their rate changed by ± 10% and about 91% of the items will have their rate changed by ± 20%. The average change in the rate, after elimination, is -12%. It appears that the demand rates for secondary items with c less than 15% are not changed substantially by the elimination of points outside the confidence belt.

(4) Items with a coefficient of variance between 15% and 30% should be closely examined for possible explanation of unusual points. Figures 9 thru 16 show examples of those items. The average change in the demand rate after elimination of unusual points is-33%.

(5) Figures 17 thru 24 show examples of items with a coefficient of variance of more than 30%. Items in this group should be thoroughly examined for a statistical and technical review. The following courses of action are suggested:

(a) Improvement may be made in the demand rate by elimination of unusual points. An example in this group is the rate for a connector

used in G254, G256 and G287. The rate before and after elimination of unusual points is 102 and 28, respectively, for 100 major items and 15 days. The reason for the high rate was an annual demand of 44,000 connectors from Fort Benning, an active training center.

(b) There is unsufficient demand data for the rate computation. By combining two years of data it may be possible to improve the rate. This may be done for items in this group. The new rate obtained had smaller coefficient of variance. (See P.R. #8)

(c) Other factors beside density are affecting the demand. Technical analysis of these rates should give an indication of the existence of these factors.

b. Reduction of demand variability

As stated previously, the density of major items is only one of the factors affecting demand variability. Eliminating the effect of densities in demand variability, an estimate can be obtained of the variability due to other factors. It is easy to show that:

$$\Sigma(di-\bar{d})^2 = \Sigma(di-\hat{b}Ui)^2 - \bar{d}\Sigma(di-\hat{b}Ui) + \Sigma(\hat{b}Ui-\bar{d})^2$$

The variability of the demands with respect to themselves is measured by the quantity on the left hand side. The first two terms in the right hand side measure roughly the variability of the demands with respect to the line of best fit. The last term measures roughly the variability explained by the line of best fit. The ratio of this latter variability to the variability of the demands will be called the percentage reduction of variability of demands due to the least square fit. A high value of this percentage will indicate a good fit, that is, most of the demand variability is due to changes in density.

Items with	$c \leq 15\%$	$15 < c \leq 30\%$	$c > 30\%$
Average percentage Reduction of Variability	70%	40%	10%

It is seen that for items with low coefficient of variance most of the variability of demand is due to changes in density.

c. Time variability of demands

By computing the demand rate of items for successive periods of time it will be possible to classify the items in four categories:

(1) Items whose demand rates do not change in time. Fluctuations observed in the demand rate will be called normal if they are within two average standard errors.

(2) Items whose demand rates increase with time

(3) Items whose demand rates decrease with time

(4) Items with fluctuating demand rates

For the sample of items considered the percentages, 46%, 18%,18% and 18%, were observed in above categories.

Demand rates for items with stable rates and with a coefficient of variation of less than 15% are ideal predictors of future demand.

Above information will be available for all active items when several years of rates have been computed.

d. Prediction of demands

Demand rates measure past utilization of an item in relation to the density of major items. If conditions other than density change substantially from year to year, a demand rate, no matter how good, will not be a good predictor of future demands.

(1) Prediction of field demands using field rates

A predicted demand for the year 1957 was formed according to:

$$D_{57} = b_{56} U_{57}$$

where b_{56} is the basic rate for 1956 and U_{57} is the density for 1957. This predicted demand was compared with the actual demand and the following ratio was obtained:

$$P = \frac{D_{57}\text{ (predicted)} \quad -D_{57}\text{ (active)}}{D_{57}\text{ (predicted)}}$$

For stable items with $c < 15\%$ the median value of P was -.05 and the range was between -.9 and .6. For all items, the median was 0.13 and the range was between -3.9 and .9. More complete information on this type of prediction will be given in Progress Report #8.

(2) Prediction of all demands

A prediction of world wide demands was attempted using a field demand rate and a projected world wide density. For purposes of comparison, information from the NICP was requested on their prediction of future requirements and the actual issues. In 48% of the cases both predictions were under and in 19% of the cases both were over the actual issues. In general, it can be said that predictions of issues based on field demand rates are lower than prediction of issues based on NICP methods of predicting.

OSAA rates are field rates and should be used to predict only field demand. In order to obtain a prediction of all the demands, to the field demands based on field rates, the following demands should be added: rebuild demands based on rebuild rates and extra demands generated at depot level. It is hoped that Progress Report #8 will throw some light into this method of prediction.

e. Use of a Poisson Model for Rate Computation

If demands behave in a Poisson fashion then the square root of demands should fit linearly to the square root of the density.*

A fit of this type was made for approximately 60 items and a comparison was made with a fit using demands versus density. The coefficient of variance for the first fit is generally lower than the coefficient of variance for the second fit. However, when compared with future demands, the "square root fit" underpredicts even more than the demand versus density fit. (See Progress Report #8)

f. Future Plans

(1) Use of residuals to improve prediction

To improve predictions, the following method is being tested: $D_{57} = a_{56} U_{57} + c_{56} r_{56}$

where a_{56} and c_{56} are coefficients obtained by least squares using 1956 data. U_{57} is the projected density for 1957 and r_{56} are the residuals left unexplained from 1956 fit. Some improvements had been observed in this method of prediction. More results will be shown in Progress Report #8.

(2) Model accounting for a possible variation of the standard error in terms of the density

The formulas illustrated in Exhibits 1 and 2 are based on a constant standard error. It if possible to account for a change of standard error in terms of the density. Two models are being tested: a) standard error changes linearly with the density; b) standard error changes according to the square root of the density. In the first case the basic rate is:

$$\text{(a)} \quad b_1 = \frac{\Sigma \dfrac{d_i}{u_i}}{n}$$

and in the second case the basic rate is:

$$\text{(b)} \quad b_2 = \frac{\Sigma\, d_i u_i}{u_i}$$

* Suggestion made by Dr. J. Tukey of Princeton University

When b_1 and b_2 are compared with the b obtained on the assumption of constant standard error,

$$(c) \quad b = \frac{\Sigma d_i u_i}{\Sigma u_i^2}$$

it is seen that the basic demand rate is an average value for the rate of each reporting unit with the following weights used for each reported demand:

for b weight U^2

for b_1 weight U

for b_2 weight 1

In fact, the three formulas can be written explicitly as follows:

$$b = \frac{\Sigma \frac{d_i}{u_i} \cdot u_i^2}{\Sigma u_i\ 2} \qquad b_2 = \frac{\Sigma \frac{d_i}{u_i} \cdot 1}{n}$$

$$b_1 = \frac{\Sigma \frac{d_i}{u_i} \cdot u_i 2}{\Sigma u_i 2}$$

It is thus seen that a rate based on d versus U with constant standard error is affected more than the other rates by points with high demand at high density or low demands at low densities.

(3) Effect of fringe demands in the demand rate

Some demand rates are quite low due to the effect of very low demand at high density. It is possible that some of these demands were not accounted for, due to their being "locally fringe demands". The effect of these demands are presently under consideration. By incorporating all the demands reported in 1546 Forms in the computation of rates, it will be possible to determine the importance of these extra demands. The task is very difficult because of the large number of forms that have to be researched and because these demands are not in IBM punched card form.

19. Technical Analysis of Rates. The most important function of OSAA is the technical analysis of demand rates for maintenance and procurement purposes. The ultimate purpose of this research is to attempt a classification of the secondary items in families, using perhaps dollar value as a yardstick, according to the categories found to be of significance in demand variability. A family may include items that are interchangeable, exhausted to or from another item. Considerations of the material of which the part is made, function that the part performs, location of the part in the major item, and policy about mandatory replacement may play an important role in this classification.

K = 500	U	u	d
01044-3	305	1	
01088-3	523	1	42
02085-6	123	0	
03006-4	981	2	131
04013-6	354	1	1
04142-6	334	1	76
04225-6	129	0	
04351-6	328	1	199
05010-5	3557	7	1234
07005-2			
09038-3	2080	4	1427
09057-3	700	1	26
09076-3	962	2	
11074-5	436	1	111
11814-1			
14040-5	4813	10	471
15014-2	3102	6	559
15056-3	3483	7	305
16027-4	531	1	124
18043-2	2692	5	165
19035-1	2130	4	258
20051-5	476	1	
20106-5			
23037-5	1330	3	170
28013-1	188	0	
28043-1			
29040-4	206	0	
30113-1	572	1	
30122-1	800	2	37
30149-1			
30319-1	734	1	
31001-3	9564	19	1506
33058-2	368	1	
34031-4	2271	5	356
36030-2	875	2	
38042-3	285	1	41
41014-4	2001	4	72
41093-4	891	2	179
41133-4	323	1	
41184-4	66	0	
44008-2	4157	8	258
44019-2	2391	5	100
44028-2			
44036-2	123	0	
44055-2	521	1	
45016-6	6135	12	333
45171-6	409	1	143
N = 41	62289	125	8324

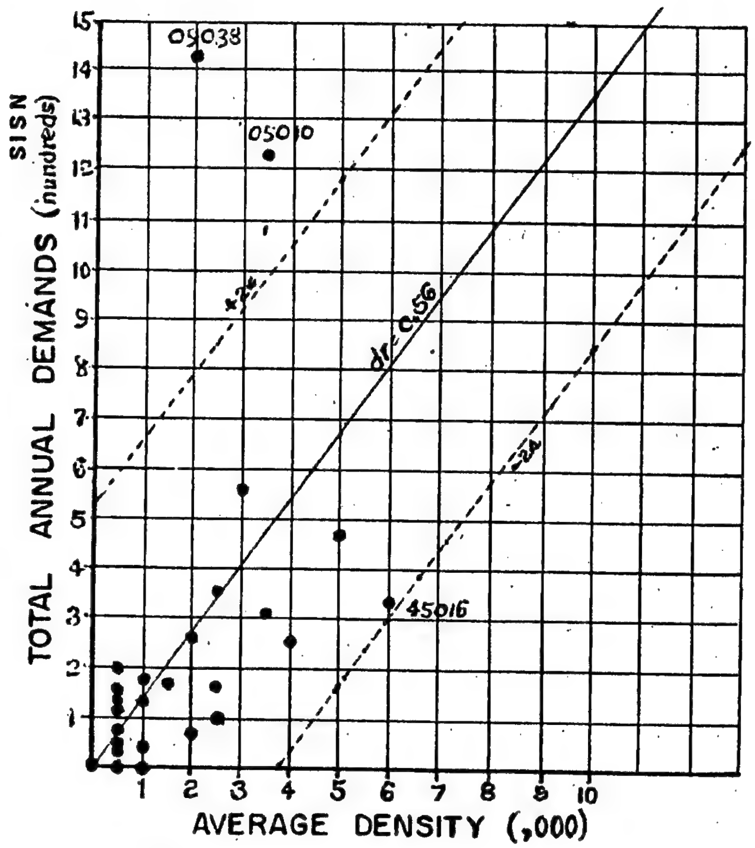
Single Application Item

SPRING SISN 1005-716-1839

$b = \frac{\Sigma ud}{\Sigma u^2} = 67.61$ $\quad s = \sqrt{\frac{\Sigma (d-bu)^2}{N-1}} = 255.25$

$s_b = \frac{s}{\sqrt{\Sigma u^2}} = 8.19$ $\quad C = \frac{s_b}{b} = 12\%$

$DR = b \cdot \frac{100}{K} \cdot \frac{1}{24} = .56$ $\quad s_{DR} = s_b \cdot \frac{100}{K} \cdot \frac{1}{24} = .01$



MISN = 1005-670-7675
Carbine, cal .30 M2

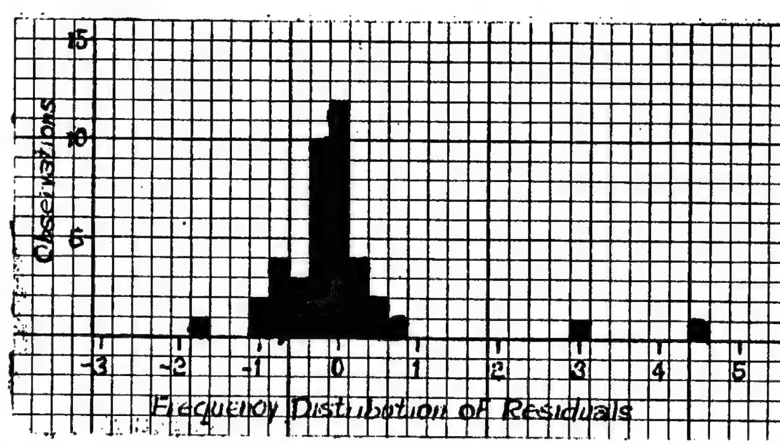


Exhibit 1

Q per MI	1	2	$1(U_1)+2(U_2)$		
MI applic.	F342 U_1	Y004-2 U_2	U	u	d
01044-3					
01088-3					
02085-6	8		8	2	
03006-4					
04013-6	1		1	0	
04142-6	5	12	29	6	144
04225-6	5	16	37	7	90
04351-6					
05010-5					
07005-2					
09038-3					
09057-3	5		5	1	66
09076-3	3		3	1	
11074-5	12	30	72	14	311
11814-1					
14040-5					
15014-2					
15056-3					
16027-4					
18043-2	27	12	51	10	512
19035-1	6	20	46	9	244
20051-5	1	13	27	5	188
20106-5					
23037-5					
28013-1					
28043-1	6		6	1	
29040-4					
30113-1	4	18	39	8	265
30122-1	13	11	34	7	199
30149-1	5	12	29	6	261
30319-1	6	12	30	6	192
31001-3					
33058-2	9	12	33	7	203
34031-4					
36030-2		14	28	6	232
38042-3					
41014-4	88	44	175	35	1851
41093-4					
41133-4					
41184-4					
44008-2		4	8	2	100
44019-2					
44028-2		8	16	3	43
44036-2	4	4	12	2	36
44055-2					
45016-6	1	12	25	5	44
45171-6		6	12	2	380
K = 5 N = 23	208	259	726	145	5361

Multiple Application Item

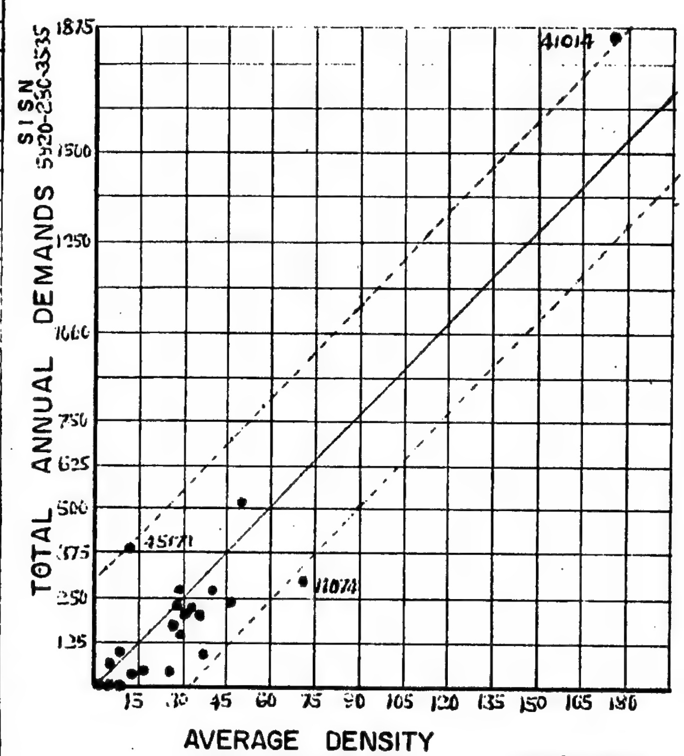
FUSE, Cartridge 5amp 25v SISN 5920-280-3535

$$b = \frac{\Sigma ud}{\Sigma u^2} = 44.14 \quad s = \sqrt{\frac{\Sigma(d-bu)^2}{N-1}} = 141 \quad s_b = \frac{s}{\sqrt{\Sigma u^2}} = 3.13$$

$$C = \frac{s_b}{b} = 7\%$$

$$DR_1 = b \cdot \frac{100}{K} \cdot \frac{m_1}{24} = 36.79 \quad DR_2 = b \cdot \frac{100}{K} \cdot \frac{m_2}{24} = 73.54$$

$$s_{DR_1} = s_b \cdot \frac{100}{K} \cdot \frac{m_1}{24} = 2.61 \quad s_{DR_2} = s_b \cdot \frac{100}{K} \cdot \frac{m_2}{24} = 5.22$$



F342 FC system T33 1230-692-0950
MISN = Y004-2 Director sta AN/MSA-7 1430-568-8174

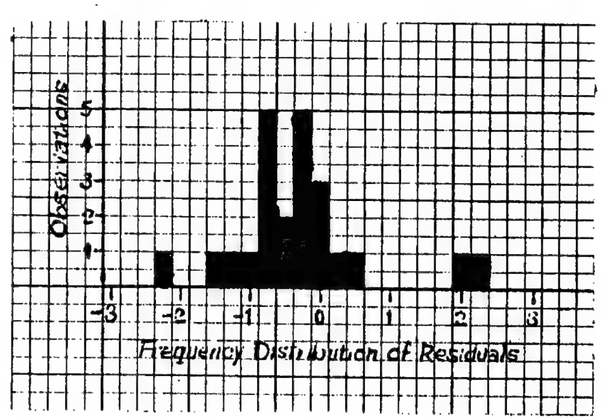
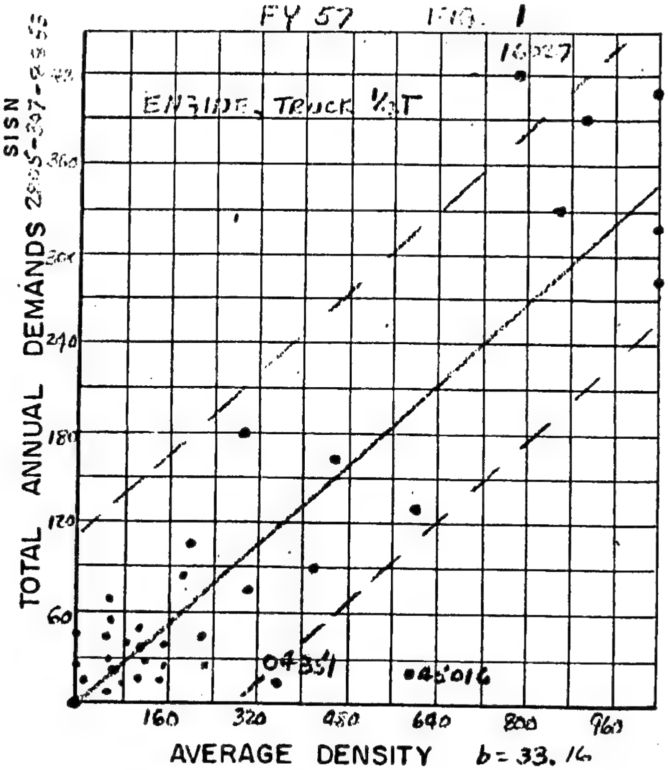


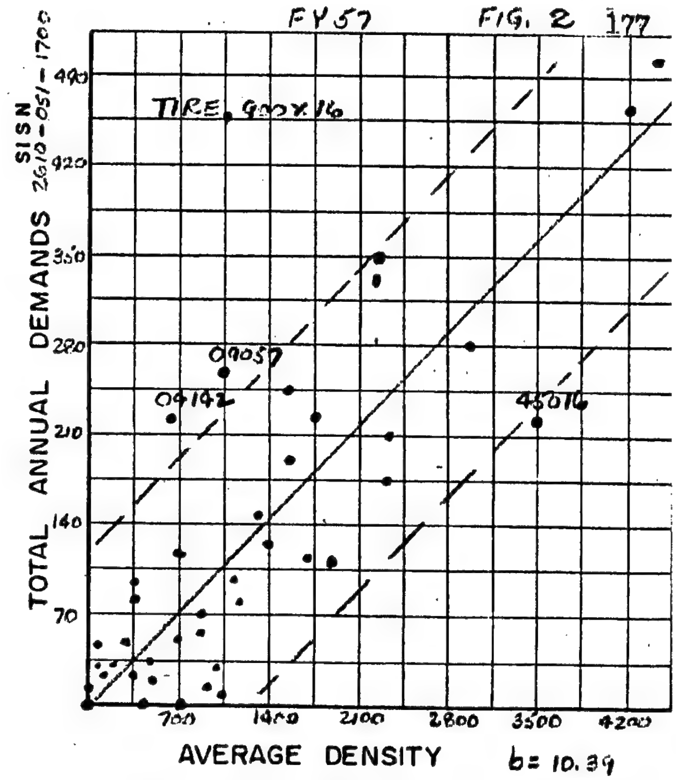
Exhibit 2

FY 57 FIG. 1



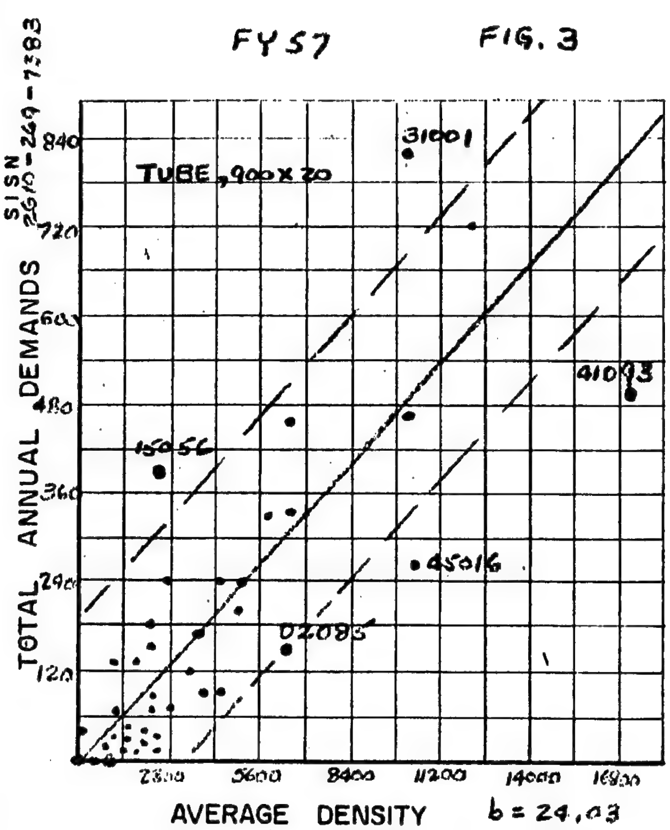
b = 33.16
MISN = SNL = G 740
a = 54.14
DR = 1.39
C = 6%
a_{DR} = 0.09

FY 57 FIG. 2



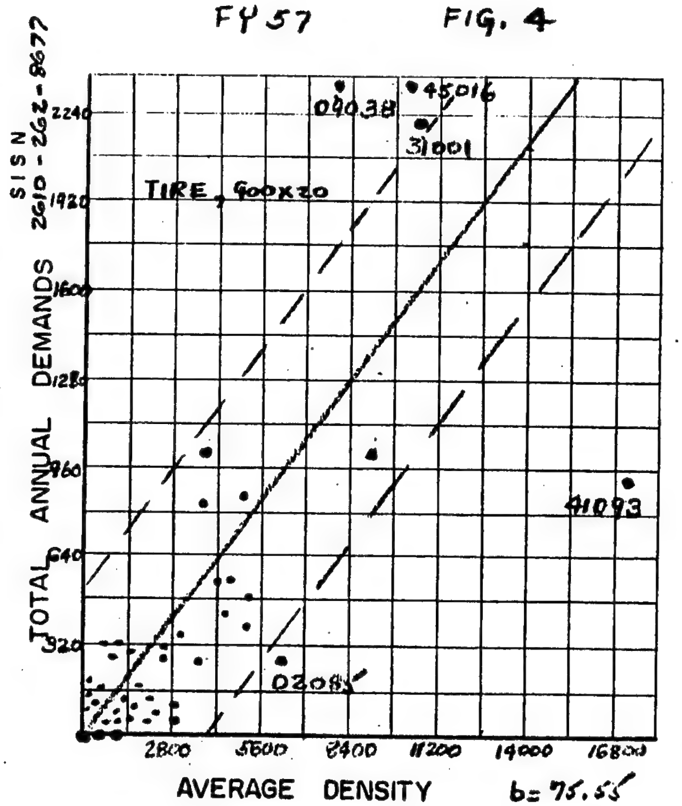
b = 10.39
MISN = SNL's = G 741
G 748
a = 60.17
DR = 0.44
C = 6%
a_{DR} = 0.03

FY 57 FIG. 3



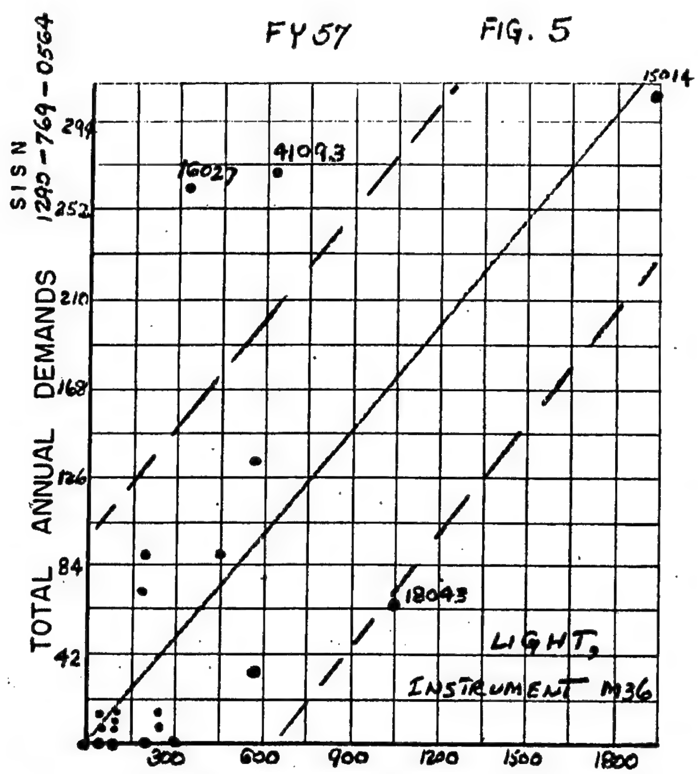
b = 24.03
MISN = SNL's = G 742
G 749
a = 76.88
G 751
DR = 0.20
C = 6%
G 756
G 754
a_{DR} = 0.01

FY 57 FIG. 4



b = 75.55
MISN = SNL's = G 749
G 742
a = 313.09
G 751
DR = 0.63
C = 7%
G 754
G 756
a_{DR} = 0.05

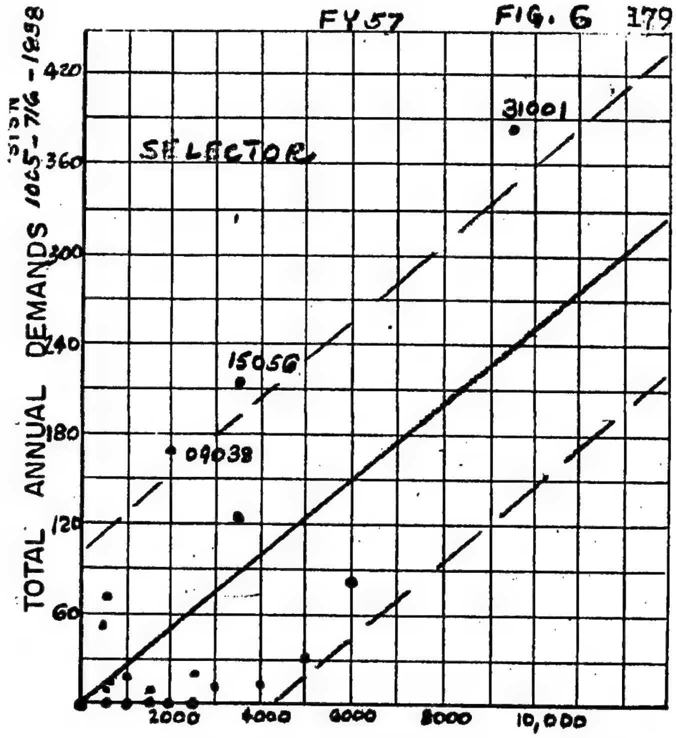
FY57 FIG. 5



b= 8.17
a = 50.74
DR= 0.68
a_{DR} =0.08
C=12%

MISN = SNL'S = C021 G199 G262 D048
C074 G232 G264 G287
C077 G244 D028 G251
D024 G254 D032 G263
D029 G256 D03?

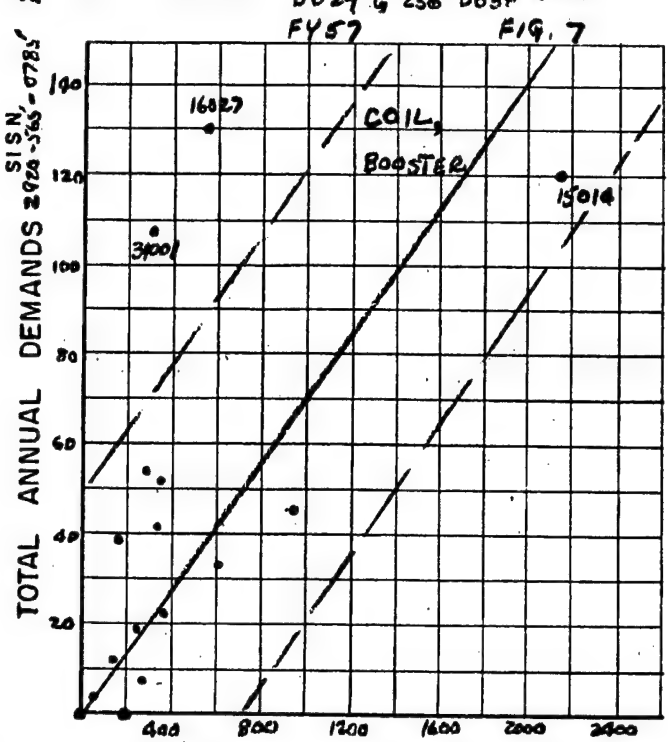
FY57 FIG. 6



b = 12.53
a = 51.03
DR = 0.10
a_{DR} = 0.01
C=13%

MISN = SNL = B 028

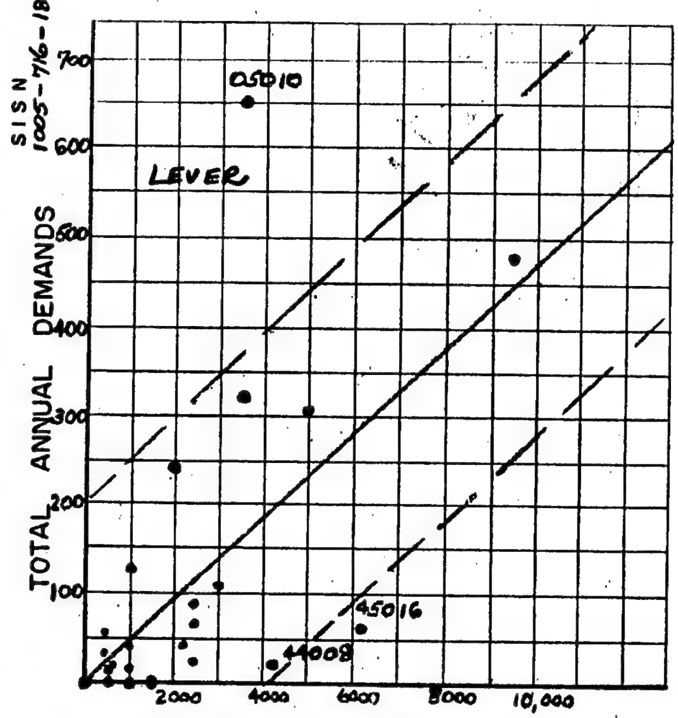
FY57 FIG. 7



b= 6.99
a = 29.52
DR = 0.59
a_{DR} = 0.08
C = 13%

MISN = SNL'S = G244 G254 G260
G251 G256 G262
G252 G258 G279
G253 G259 G287

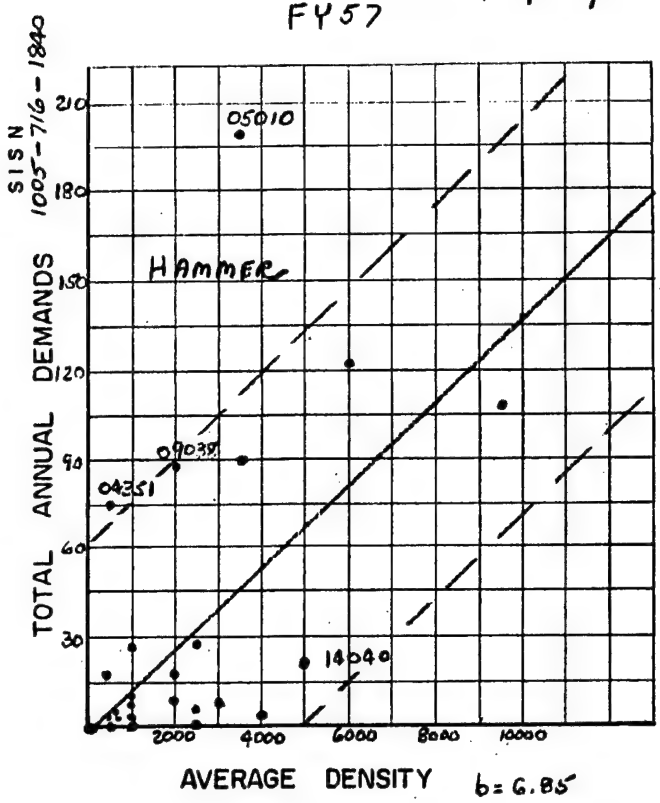
FY57 FIG. 8



b = 23.72
a = 103.44
DR = 0.20
a_{DR} = 0.03
C = 14%

MISN = SNL = B 028

FY57 FIG. 9

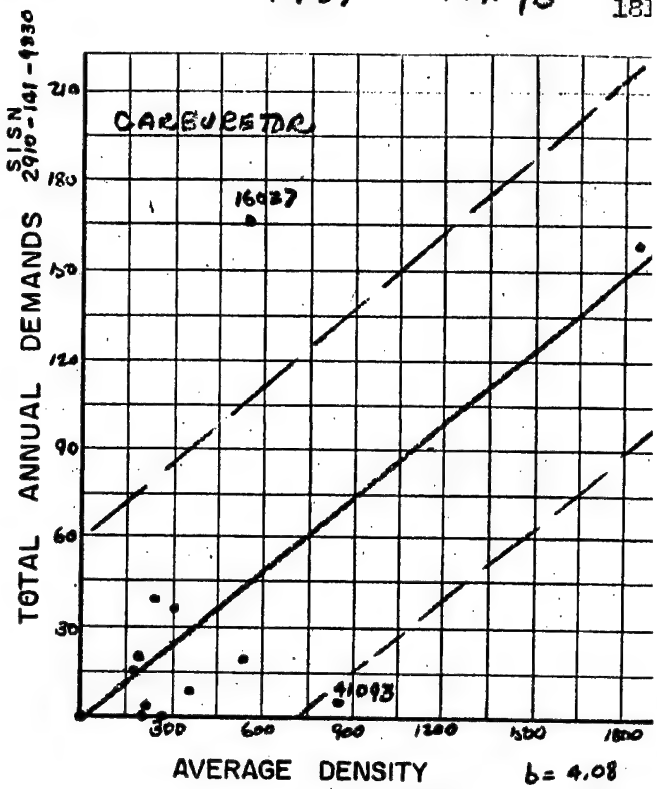


MISN = SNL = B 028

b = 6.85
a = 33.59
DR = 0.06
a_{DR} = 0.01

C : 16%

FY57 FIG. 10

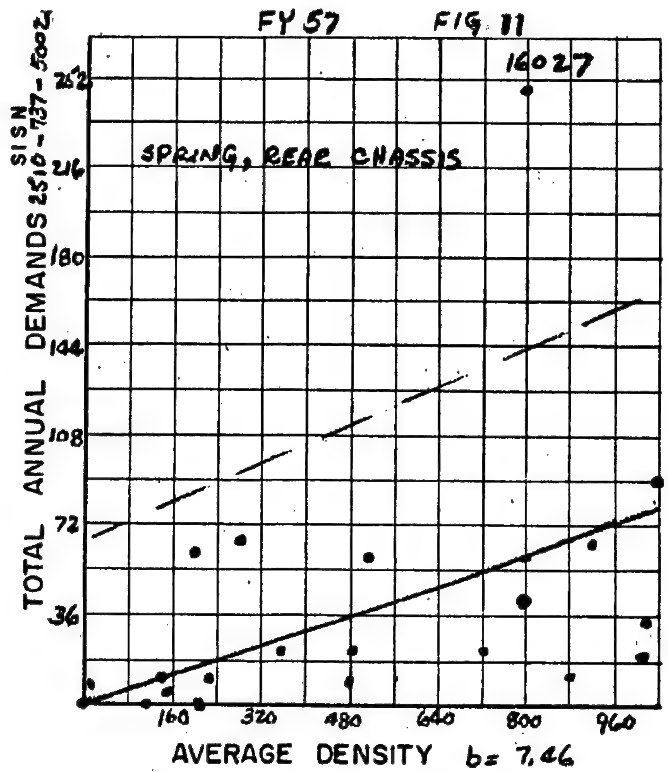


MISN = SNL'S = G 244 G 259 G 254 G 260 G 256

b = 4.08
a = 30.51
DR = 0.68
a_{DR} = 0.11

C = 16%

FY 57 FIG 11

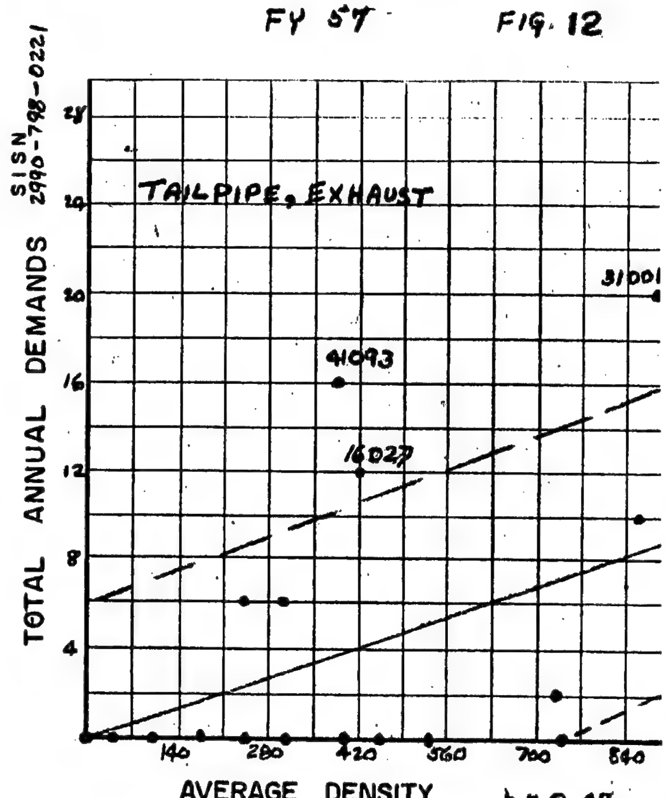


MISN = SNL = G 740

b = 7.46
a = 34.23
DE = 0.31
a_{DR} = 0.05

C = 17%

FY 57 FIG. 12

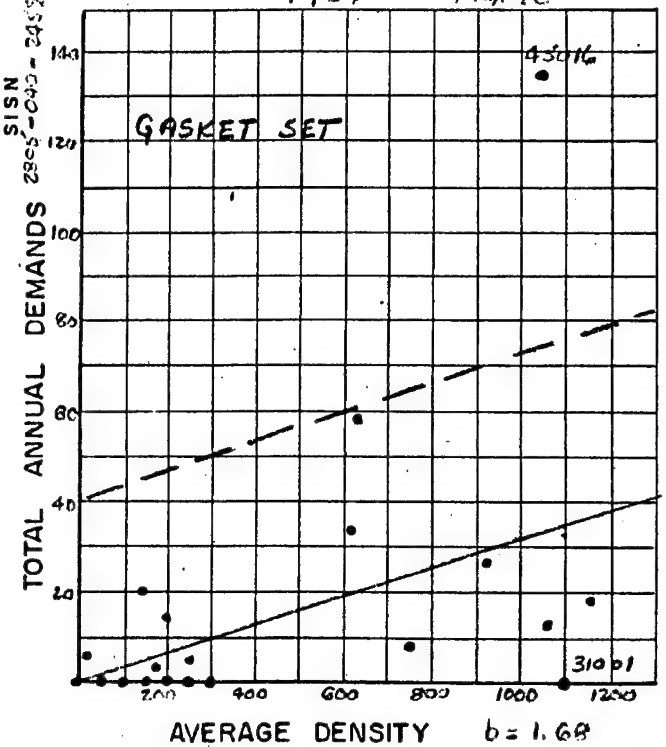


MISN = SNL = G 741

b = 0.48
a = 3.63
DR = 0.04
a_{DR} = 0.01

C = 18%

FY57 FIG. 13

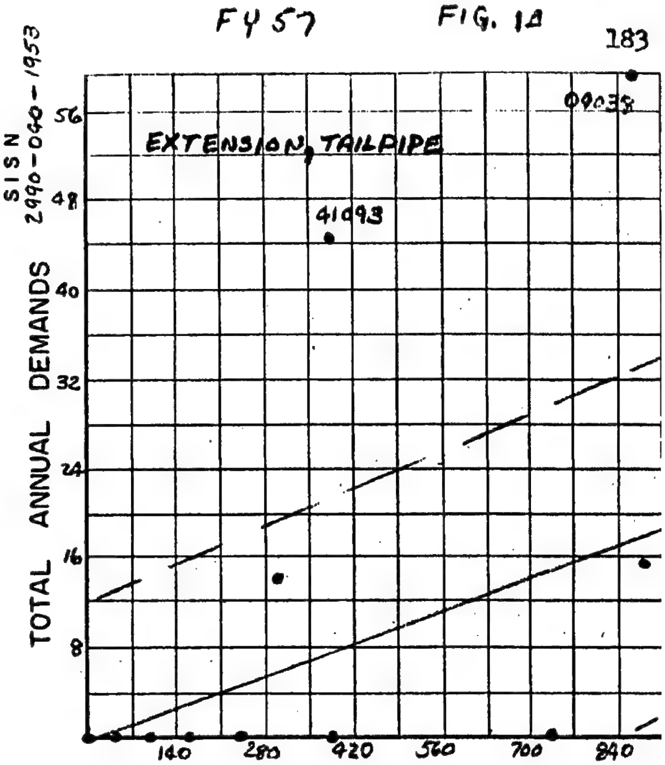


b = 1.68
a = 19.60
DR = 0.14
a_{DR} = 0.03

MISN = SNL = G 758

C = 21%

FY 57 FIG. 14

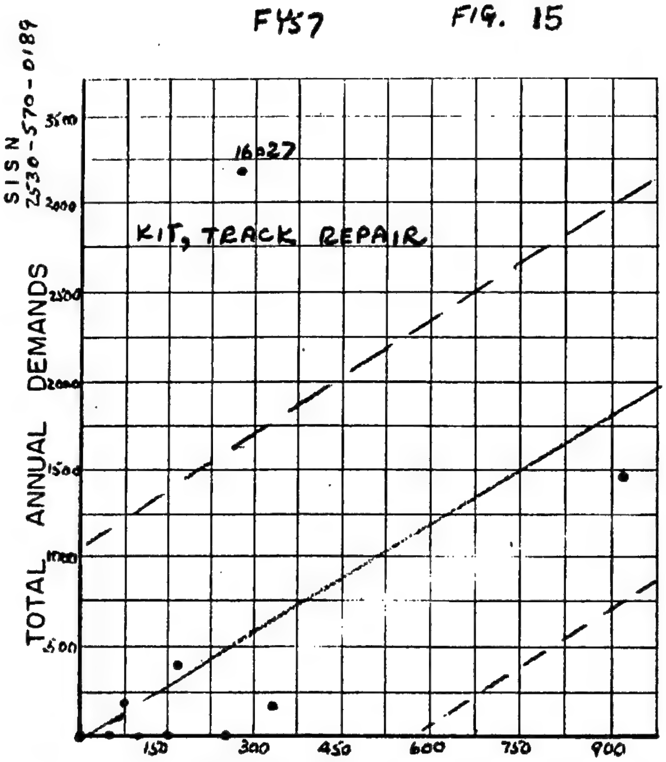


b = 0.10
a = 10.14
DR = 0.08
a_{DR} = 0.02

MISN = SNL = G 741

C = 24%

FY57 FIG. 15

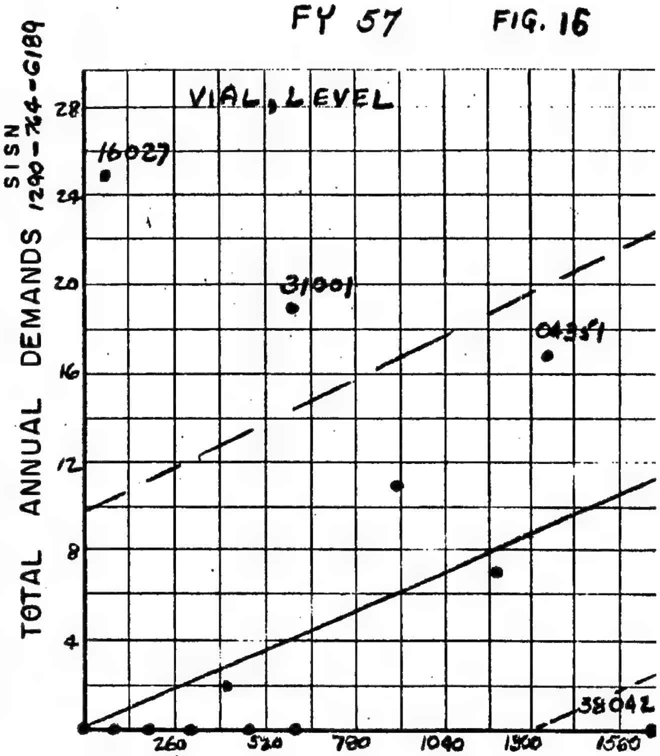


b = 101.78
a = 578.26
DR = 8.45
a_{DR} = 2.20

MISN = SNL'S = G 244, G 254, G 256, G 259, G 262

C = 26%

FY 57 FIG. 16

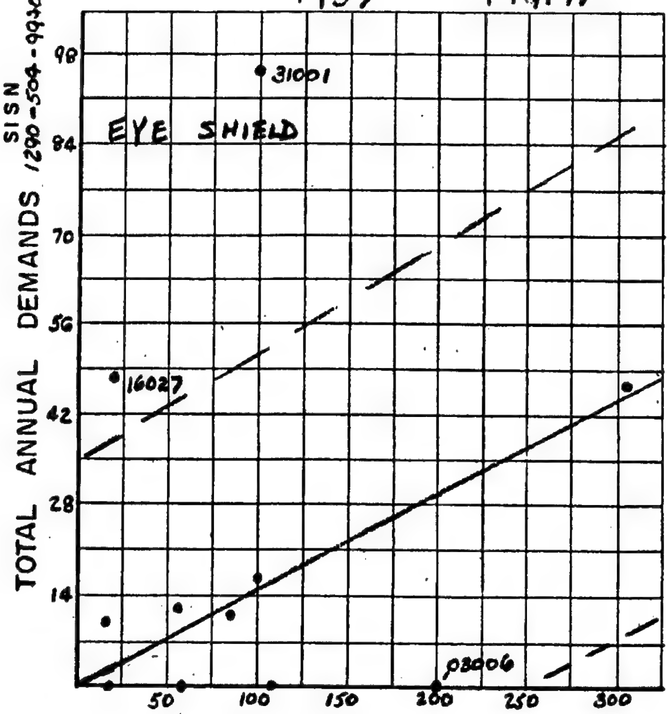


b = 0.66
a = 5.39
DR = 0.03
a_{DR} = 0.01

MISN = SNL'S = A 033, A 043, A 082, G 102, A 070, F 015

C = 27%

FY57 FIG. 17

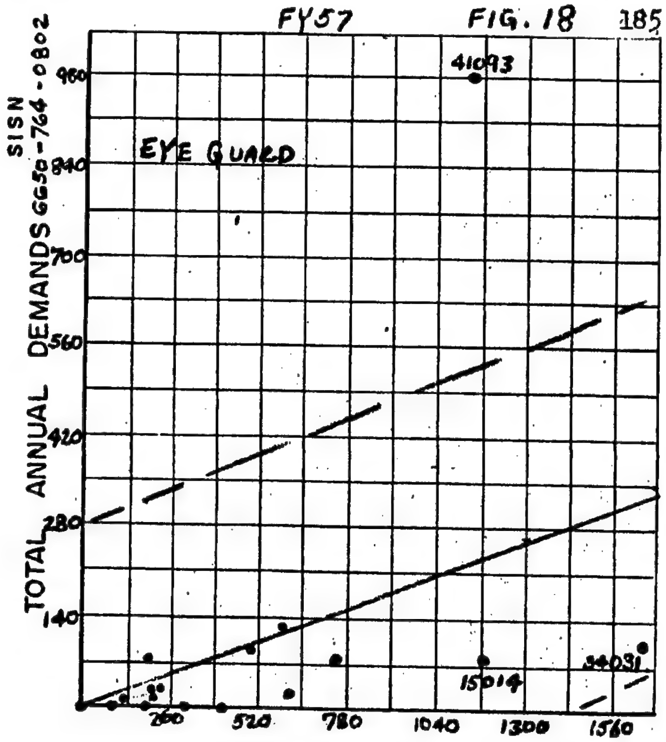


b = 2.94
a = 17.75
DR = 0.61
a DR = 0.16

MISN = SNL'S = G 238, G 279, C 021, C 039, D 024, D 031, D 065

C = 27%

FY57 FIG. 18

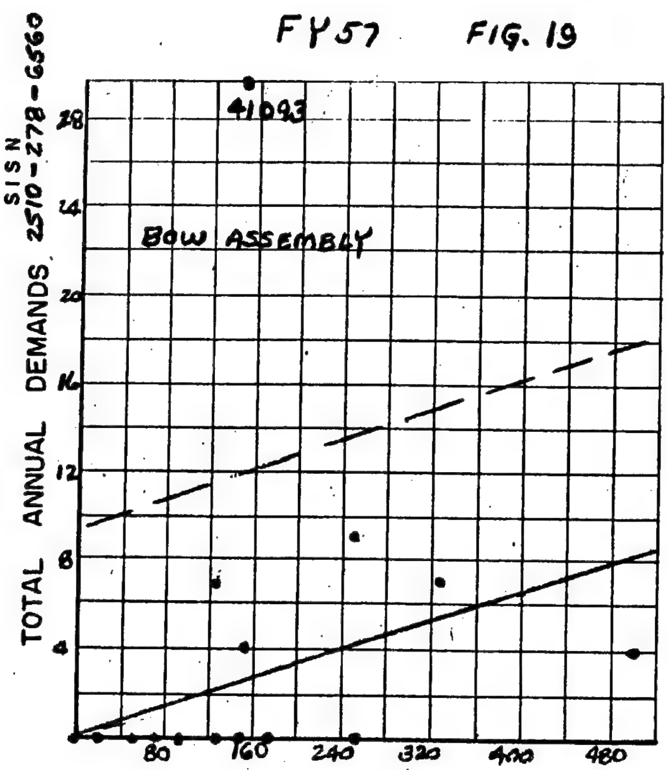


b = 20.38
a = 146.95
DR = 0.86
a DR = 0.25

MISN = SNL = F 238

C = 29%

FY57 FIG. 19

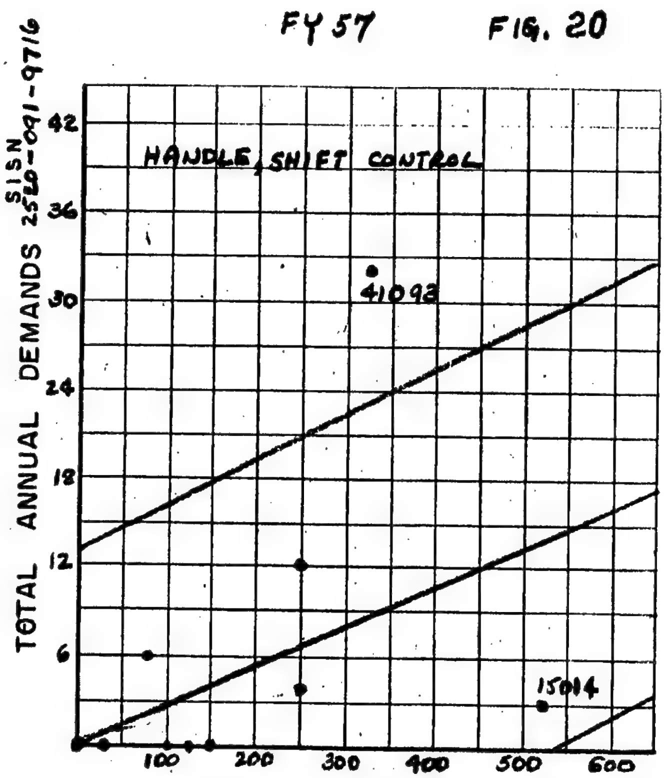


b = 0.41
a = 4.59
DR = 0.07
a DR = 0.02

MISN = SNL = G 748

C = 32%

FY 57 FIG. 20

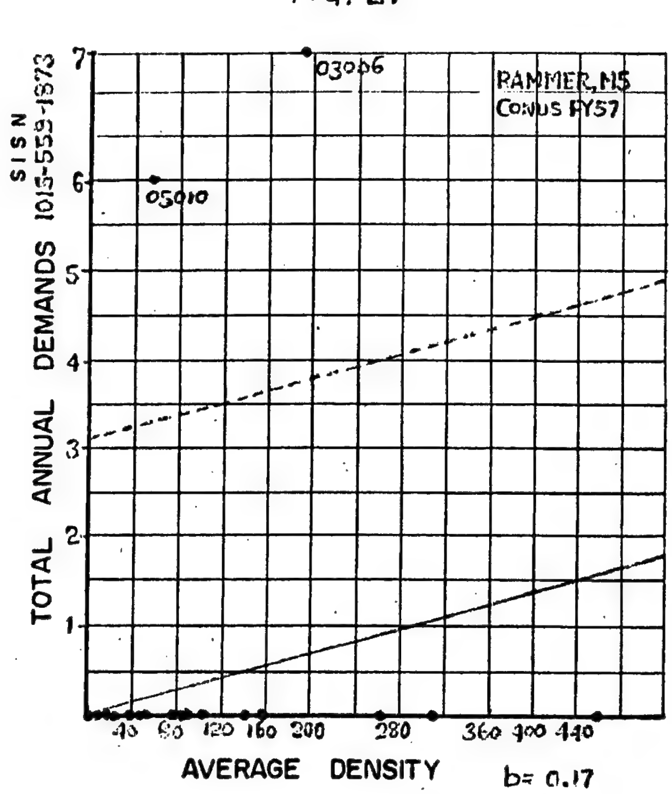


b = 0.66
a = 6.71
DR = 0.11
a DR = 0.04

MISN = SNL'S = G 254, G 256

C = 32%

FIG. 21

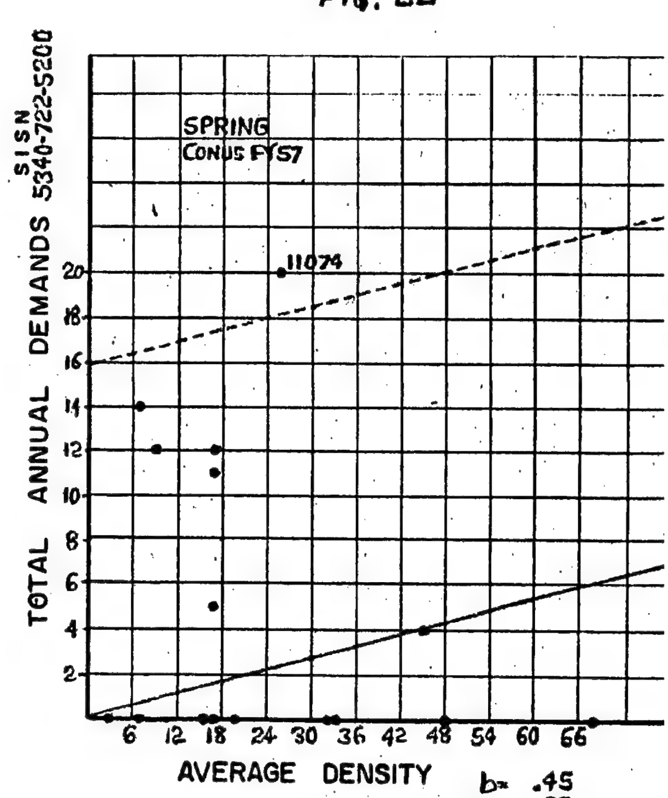


MISN = SNL C021, C077, G199, G238, G258

b= 0.17
Δ= .11
DR= .01
ΔDR= .009

C=65%

FIG. 22

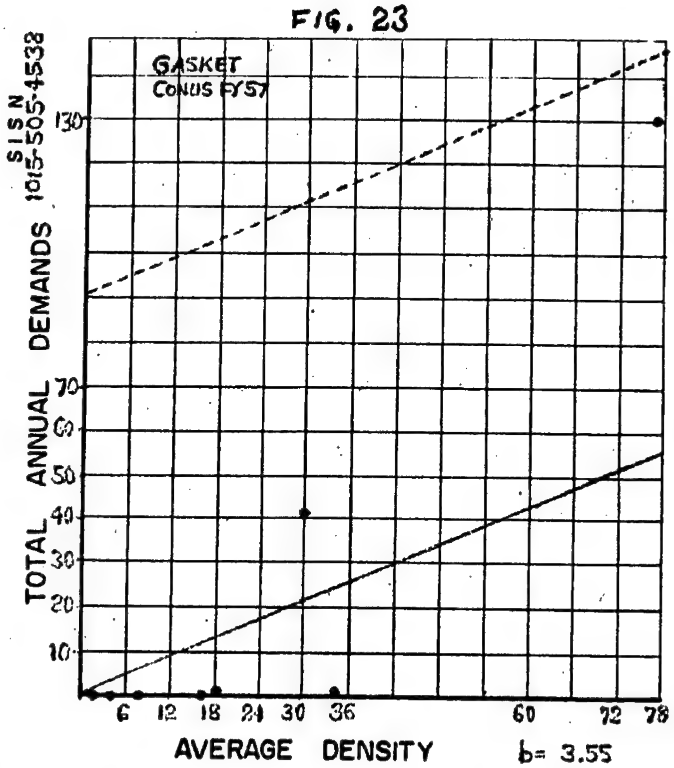


MISN = SNL D038, G244

b= .45
Δ= .33
DR= .37
ΔDR= .27

C=73%

FIG. 23

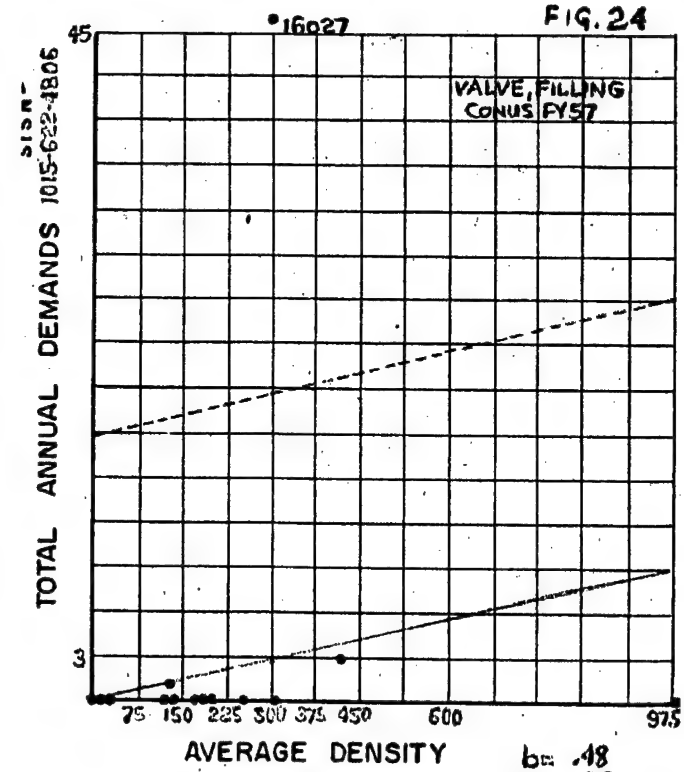


MISN = SNL D048

b= 3.55
Δ= 2.15
DR= 2.95
ΔDR= 2.04

C=69%

FIG. 24



MISN = SNL G238, G251, G254, G256, G258, G259, G262, G279

b= .48
Δ= .36
DR= .04
ΔDR= .03

C=75%

A COMPUTER SIMULATION OF SMALL COMBAT ACTIONS

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The AUTOTAG computer simulation has been designed to evaluate antiarmor weapons and weapon system performance. This simulation provides the means to accomplish a multivariable systems comparison in a relatively short time.

This paper will describe the AUTOTAG program in some detail and discuss a few of the problem areas, related to antitank weapon systems, where this simulation may find application. A short discussion of a tentative plan for initial use of the simulation will also be discussed.

BACKGROUND. I would like to preface this discussion by providing a few background details related to simulation development.

About five years ago, a large unit war game was developed at CORG. This game, called SYNTAC, involves combat between large size forces and requires extensive military player participation. Umpires serve as the game control function by deciding on action outcome throughout each game. SYNTAC continues to provide information of tactical and logistical value related to large scale combat, however, this war game does not, in general, provide detailed information related to small unit combat actions.

Consequently, in order to investigate particular aspects of armor combat in greater detail, shortly after the development of SYNTAC, effort was devoted to the design of a small unit war game, TABWAG (Tank Battle War Game). TABWAG was specifically developed to investigate the antitank effectiveness of weapon systems. Play took place on a large scale terrain map with players representing each force. A single umpire was required to act as the decision maker.

TABWAG was in fact useful for small unit system evaluation, however, one of the primary limitations of this game was that playing time for a single play required about 25 man hours. Such a limitation seriously curtailed the number of situations that could be investigated using this model.

In order to reduce playing time and eliminate the need for player-umpire participation, a rigid war game, TAG(rigid in the sense that decision criteria and rules of play were built into the model), was subsequently designed. The stylized structure of TAG made it possible for an individual unversed in military doctrine to play through a combat situation.

The TAG game, or more precisely, simulation, was used in a comparative antitank weapon evaluation where a factorial design of five variables at two levels each was employed. Fifteen replications of each of the 32 combinations were played and non-parametric methods of analysis, as well as analysis of variance used on several measures of system performance. A more complete write-up of the results of this study as well as a discussion of the TAG model may be found in CORG Report R-71.

The play of a single TAG game required about one half a man hour. This was of course a distinct improvement timewise over TABWAG, however, the time required for play of a single game was still excessive for hand play of larger system comparisons. Unfortunately, the limited scope of the TAG analysis pointed out a need for larger designs involving more variables.

It was apparent that in order to broaden the scope of future analysis, playing time must be further reduced. It was also obvious that the only feasible way to accomplish this time reduction was to goto automation.

AUTOTAG has been the next step in the developmental cycle. The AUTOTAG program is an automated and more complex counterpart of TAG. The computer model was developed and subsequently programmed for the IBM 704 at the National Bureau of Standards in Washington, D. C.

AUTOTAG PROGRAM. The AUTOTAG program is an internally stored program composed of some 5000 instructions and data words. The structure of the program is such that it may be subdivided into four parts for ease of discussion. These are: Input, Simulation, Output, and Analysis. I would like to discuss each of these parts briefly.

Inputs

AUTOTAG inputs may be subdivided into four basic categories as shown in Figure 1:

1. Weapon type parameters
2. Weapon environmental parameters
3. Weapon change parameters
4. Target Selection rules

Figure 1.

The first of these, weapon type parameters, define the weapon in terms of inherent capabilities and limitations. Examples of the parameters that must be specified for each weapon class or type are illustrated in Figure 2. AUTOTAG has the capacity to handle 10 weapon types, and these parameters shown would be input for each weapon type to be gamed. Horizontal and vertical dispersions are range dependent, kill probabilities are based on weapon-target combinations, and performance times exemplified by load or obscuration time are stored as cumulative distributions, and are in turn randomly sampled when performance time values are required.

WEAPON TYPE PARAMETERS

- Horizontal and Vertical Dispersions
- Conditional Kill Probabilities
- Maximum and Minimum Ranges
- Dimensions as a Target
- Position Disclosure Characteristics
- Performance Times

Figure 2.

The second category of inputs is comprised of parameters which serve to describe the status and role of each weapon participating in the simulation. Examples of these parameters are shown in Figure 3.

WEAPON ENVIRONMENTAL PARAMETERS

- Side
- Type
- Initial Position
- Velocity
- Status as a Weapon
- Status as a Target
- Target Selection Rules
- Field of Fire
- Artillery and Mine Vulnerability

Figure 3.

Up to 31 weapons on either side may be considered. For each of these weapons an initial position and velocity are specified with respect to a stored grid network. Weapon and target status as well as field of fire are used to introduce terrain effects. As examples, terrain masking can be depicted in terms of a field of fire limitation, or an unavailable target status for a weapon would correspond to the weapon being completely concealed from all members of the opposing force.

The third category of inputs, weapon change parameters, permits modification or change of any parameter shown here, during the course of play. The use of such changes can provide a change in a weapon's position, a change in a weapon's rate of advance, or a change in a weapon's target selection rules as examples. Up to 128 of these environmental changes may be executed at specified times during the simulation.

The fourth category of inputs, target selection rules, are used to describe a weapon's fire doctrine. A primary and secondary target selection rule index may be prescribed for each weapon -- each index in turn refers to one of 128 selection rules which comprise the fourth category of inputs.

Although the discussion of inputs has by no means been complete, it should be apparent from the illustrative examples that a variety of weapon systems and system environments may be defined.

Simulation

I would like to briefly indicate, in a rather elementary fashion, how AUTOTAG interprets input data.

Each weapon participating in the simulation can be in one of several actionable states, such as: looking for a target, preparing to fire, firing, or obscured. The current and future action times associated with each weapon are stored in an action file. Game or simulation time is the control function, and I might add, also defines the length of any AUTOTAG game. Program control sorts through the action file and attempts to select any action time which is equal to current game time. If no action is found, game time is advanced by one unit and program control resorts over all action times. Assume that the action associated with a selected time is a LOOK. A LOOK action implies that the weapon is prepared to acquire a target. If the target is available and may be selected on the basis of parameters prescribed by weapon environmental inputs, more particularly, weapon target selection rules, field of fire and weapon target status, such a selection occurs. A fire time is established for the weapon on the basis of the weapon's performance time distribution. This time is then stored in the action file. The program resorts over all action times and selects the next action time in chronological order. Eventually, the fire time which has been established for the weapon will be selected by program control, unless of course an intervening action has destroyed the weapon. Assuming such an intervening action did not occur, the fire action will be performed. A computation of the kill probability associated with the fire action occurs based on weapon type inputs such as weapon dispersions as a function of weapon-target range, target size and conditional kill probability. The result of the action is stored, the weapon goes through an obscuration phase, and at a subsequent action time is prepared to attempt to acquire the same or an alternate target for a new fire action.

Performance times associated with weapon actions such as load time, and lay and fire time, are established by random sampling from appropriate performance time distributions. Fire action outcome is determined by using a random number generation technique and testing the generated number against the computed kill probability value.

I would like to touch on one additional simulation capability before closing this portion of the discussion. The AUTOTAG program is able to simulate artillery and mine action. This is accomplished through the specification of artillery or mine zones. Each zone is identified in terms of square area and rate of artillery concentration or number of mines in the zone. When a weapon enters an artillery or mine zone, a stored weapon change parameter indicates this fact to program control. This, in turn, initiates an artillery or mine zone computation. This determines if the weapon sustains a death in the respective zone prior to maximum game time (which defines the end of the game). If such a death occurs, the death action is stored in the action file and will occur at the computed time. If, however, the weapon leaves the zone, in the case of artillery, or if the weapon changes velocity or leaves the zone, in the case of mines, at a game time prior to computed time of weapon death, the action will be deleted.

Output

Output from the AUTOTAG simulation is in the form of an action by action record of all important events which occurred during the simulation. This record or log provides a detailed chronological listing of action entries, and indicates the weapon and target involved in each case. The results are in terms which require little or no translation into realistic measures of effectiveness, for example: rounds fired, kills inflicted, or deaths sustained.

Analysis

During the programming phase of the AUTOTAG project, a good deal of effort was devoted to the development of a data processing routing which would simplify a portion of the statistical problems associated with AUTOTAG output. This effort resulted in an analysis program which can count various log entries such as rounds fired by a weapon system. These counts are automatically compiled for all replays of a given situation and then average values are computed together with the variance associated with these average values. Additional statistical processing of output data, for example, analysis of variance, has already been programmed and will be accomplished on a smaller computer.

Problem Areas

I believe it would be worthwhile to indicate a few of the problem areas where it is anticipated that this model will contribute significantly in its initial application.

PROBLEM AREAS

- Composition of Antitank systems
- Parameters specified in QMR's and MC's
- Influence of weapon parameters on tactics
- Factors for war gaming
- Improvement of evaluation techniques

Figure 4

As Figure 4 indicates, the first problem area involves composition of weapon systems. For example, what combination of conventional and missile type weapons will provide optimal combat effectiveness? Problem two: What can we say about parameters specified in qualitative materiel requirements or military characteristics? What is the relative importance of and the relative trade off among these parameters? Problem three: What are the influence of weapon parameters on tactics, deployment and other factors associated with combat? Is it possible that through a comprehensive analysis of the effect of these factors we will be able to more adequately determine system requirements? Problem four: What are appropriate factors for war gaming?

Can we improve war gaming methods through a detailed study of factor importance and factor effect? Problem five: Finally, how can we improve current simulation and analytical evaluation technique? What effect will AUTOTAG output have on the design of subsequent simulations or mathematical models?

Although this is by no means an exhaustive listing of problem areas, it is sufficient to illustrate the type of problems associated with weapons system evaluation that currently exist.

TENTATIVE PLAN OF EVALUATION. The initial exploitation of the AUTOTAG model should encompass as many of these problem areas as possible, and since a multivariable evaluation is desired, economic criteria dictate the selection of some systematic method of problem attack. A rather thorough consideration of our specific needs has led us to tentatively select a factorial design approach. The design that has been chosen involves twelve factors, or variables, at two levels. Figure 5 indicates the variables which might be considered in this design.

- Quantity of primary weapon
- Type of primary weapon
- Quantity of support
- Type of support
- Terrain
- Attacker tactic
- Quantity Attacker
- Attacker movement
- Defender deployment
- Range
- Defender fire plan
- Visibility

Figure 5.

As I have indicated, each variable will have two levels; for example, one level of range may be 600 meters and the second level, 1200 meters. An effort will be made to choose variable levels such that a pronounced difference in level values will exist. If this is not done, variables that are highly significant in terms of main effect may not be detected because of inherent simulation statistical fluctuation.

A complete two to the twelfth factorial design would involve about 4100 different situations, and if each situation is to be replayed 20 to 30 times to obtain validity, the cost of carrying out such a design would be prohibitive. It is possible, however, to fractionate a factorial design substantially and still obtain data of value. * In the case of the two to the twelfth factorial design, fractionation will result in some data loss, however, it will still be possible to measure the main effect of each variable as well as 90% of all two factor interactions. The remaining two factor and higher order interactions are not measurable. It should also be pointed out that the measurable effects are not as "pure" as we would like in that they are actually confounded with higher order interactions.

Output data from the design will be handled with a variety of statistical treatments including conventional analysis of variance technique and, perhaps, factor analysis.

Subsequent designs may consider variables at three levels. For example, it is anticipated that a 1/27 fraction of a 7 variable, 3 level, factorial design will be programmed, in order to learn more about variables which had a strong effect in the initial two to the twelfth design. In this instance, where three levels of a variable are considered, other statistical treatments such as regression analysis or factor analysis may be used in conjunction with analysis of variance technique.

<u>SUMMARY</u>. In summary, I would like to stress a few important points. AUTOTAG is a highly versatile program capable of reproducing a variety of combat situations. Any plan for program utilization must be oriented towards pertinent and timely problem areas and provide a sound basis for subsequent exploitation. It is felt that this tentative plan for exploitation fulfills these criteria and provides an excellent balance between potential results and capital expenditure.

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* The National Bureau of Standards has done extensive work on this subject and has published an atlas which details optimized procedures for fractionating two level factorial designs.

ARMY LOGISTICS RESEARCH

Lt. Colonel H. W. Rice
Research Section, Plans Division,
Deputy Chief of Staff for Logistics

The purpose of my discussion is to acquaint you with the Army's logistics operations research activities. I will cover the role which operations research plays in logistic activities, and will give a resumé of the logistic operations research program objectives, policies, organizations, problem areas and future study programs.

What is logistics? We use the term loosely. It is as old as organized warfare. It is both an art and a science. Past military operations have demonstrated that without adequate consideration for logistics, it spells the difference between victory and defeat. It is concerned with the establishment and support of military forces. It encompasses the harnessing of technology and production for military needs, and it provides for the proper distribution of the materiel means of warfare to the combatant forces.

Today the gross materiel requirements for military forces has put such a demand on the nations economy, that their magnitude is controlled largely by the impact upon the economy, rather than the nature of the enemy threat.

Consequently, in its broadest sense, logistics management today is 1) the shaping of the optimum combinations of military support forces and equipment essential to U. S. protection, but consistent with economic stability, and 2) the distribution of the proper support equipments and forces to the right places.

When viewed in terms of these requirements, and when placed in its perspective of actions necessary to determine, create, produce, distribute, and efficiently utilize the end products, logistics in a field so complex and important as to demand the application of the most sophisticated means of assistance.

Turning now to operations research, what is operations research? In simple terms, operations research is a tool which can be used by military management to assist in solving its problems. It consists of a systematic approach which seeks to define a problem, draws hypothesis toward its solutions, and provides for testing and evaluation of its conclusions.

At this conference we have heard discussed many of the scientific techniques which are a part of this management tool. These techniques such as linear programming, probability statistics and value theory make possible the establishment of new facts and relationships. They permit investigation between many facets of a problem and between the problem and other related problems.

Operations research is thus a tool which affords new dimensions to aid the logistician in the solution of his problems. Accepting this fact, what is the scope of the Army's logistical management problems which this operations research tool can help to solve?

Army logistics are global in nature. They encompass the control of annual expenditure of $4.5 million for support and equipment; the management of physical inventories valued at approximately $24 billion and the control of some 300 major installations.

In addition to these current operational responsibilities and associated problems, the Army logistician is concerned with logistical planning problems for future combat and zone of interior operations. In fact, probably the most difficult, complex and urgent problem facing the logistician is how to establish future logistics doctrine and combat support in the face of rapidly changing tactical doctrines, explosive political environments and economic pressures.

These latter factors are the resultants of three current developments which are having major military impact.

First, the revolution in science and technology is ushering in many new and complex weapons. These scientific innovations cause major changes in tactical and strategic doctrine. Second, industrially backward nations are eager to secure the benefits from modern civilization. They are vulnerable to internal political disturbances. They generate international economic political and military tensions, and thirdly, we are undergoing a period of rising costs. We are experiencing all of the frustrations which these spiralling costs produce-relationships to national economy, international economic competition, social reforms and political readjustments.

Against this panorama of major developments, the art and science of warfare are obviously changing. The operational concepts of tactics and strategy are necessarily fluid.

For the logistician, this situation of fluid tactics and new developments-scientific, social, political and economic - have created problems of a magnitude not previously encountered.

These problems are too complex for the logistician to resolve solely from past experience. These problems involve too many facts and too many facets which have not counterparts in past experience.

Logistical studies have been made in the Army for years, but today's studies require a greater need for systematic and comprehensive analysis of broader and longer range problems.

Necessarily, increased emphasis is being placed on the application of scientific techniques as an aid to logistical decision making.

The application of these scientific techniques in terms of DA logistics research studies are provided through two sources - contracts with civilian commercial or educational organizations, and through so called in-house studies performed by military organizations. The organizational control of this study effort is vested in the Chief of Research and Development, DA. The latter has cognizance of all Army research studies. This responsibility of the CR&D includes responsibility for insuring coordination, integration, avoiding duplication and assuring optimum utilization of research study results.

The Army Research Office, an integral part of OCR&D, has been established as the focal point for insuring accomplishment of an integrated research program responsive to the growing demands of the Army.

In the fields of non-materiel logistics research, the CR&D relies upon the guidance and coordination furnished thru the ODCSLOG,DA. It is through this guidance, coordination and collaborating with ODCSLOG that the direction and supervision of the Army non-materiel logistics research efforts are exercised.

Much of the detailed operational aspects of non-materiel logistics research - programming, administration, and supervision - are performed by the Army Logistics Management Center, Ft. Lee, Virginia, a field agency of DCSLOG.

The Army Logistics Management Center conducts such studies as are required to assist DCSLOG in developing guidance for future logistics organizations, systems and procedures, and to assist the CG, USCONARC in the formulation of logistics organizations and doctrine for the Theater Army. In performing these tasks, the ALMC participates in the development of the logistics research programs of the Technical Services. In addition, ALMC has been designated as the office of record for all information pertaining to past DA logistical studies, current DA studies and information pertaining to the current logistical study efforts by the Navy, AF, other government agencies and commercial organizations. I suggest that each of you having responsibilities in the logistics research area, establish the necessary liaison with personnel of the LR&D Division of ALMC, Ft. Lee, Va.

In terms of logistic study programming responsibilities, the DCSLOG is primarily concerned with those projects having Army wide logistical interest. The CG, USCONARC efforts are centered on field Army operational needs; and the Chiefs of Technical Service studies are devoted to their particular specialty fields.

The basic objective of these Army logistics research programs is to assist in the development of a logistics systems which will be responsive to the growing strategic and tactical operational requirements of the Army. These studies necessarily deal with the nature of future warfare. Since future war is a broad subject and study efforts are both expensive and time consuming, objectivity and guidance are necessary.

The basic DA regulation covering logistics research is Logistics Directive No. 136-1, ODCSLOG, subject "Logistics Management Type and Operations Research Projects" dated 27 February 1959. This regulation establishes the policies under which contractual studies will be initiated, the contract approval authorities, responsibilities for supervision of the studies and procedures for necessary study, reporting and evaluation.

In terms of monetary effort, the combined FY 59 non-materiel logistics research programs of DCSLOG, USONARC and the Technical Services amounts to approximately $3,000,000. Some of the principal contractors include ORO of Johns Hopkins University; Harbridge House, Inc; George Washington U; Melpar, Inc; Stanford Research Institute, Technical Operations Inc; MIT, Planning Research Corp and others.

The principal subject matter covered in these studies include inventory management, financial management, maintenance, transportation, gaming and control systems.

I will be the first to admit that our current program does not meet the standards which we would like. It was precisely for this reason that ALMC, LR&D Division was established in order that proper study programming and control of studies could be accomplished. At present we do not feel that the desired degree of integration of study efforts is being achieved. Adequate inventories of past studies have not been established; and proper evaluation and utilization of past efforts have not been attained. Some of these shortcomings can and are being corrected without too much difficulty. Other problem areas not so easily corrected include the following; first, it is difficult to convince operating personnel of the value of log research. It is difficult to demonstrate to them the value of objective evaluation. Considerable effort will be necessary to overcome this obstacle. Secondly, there is a lack of communication between the operating personnel (those who have problems) and the research personnel attempting to assist in the problem solution. This lack of mutual understanding appears to stem from a failure to define the problem adequately in turns of management and research personnel appreciation, and the improper selling of the merits of operations research. Lastly, there is the failure to translate the study findings into operational use. This is a corollary of the failure to communicate between the operational and research personnel. Many of our past studies were published and placed in some file. There has been little effort to translate these studies into practical applications. Nothing will kill a research program so fast as a failure to provide a payoff. Consequently our policy on new studies is weighed heavily in terms of practical applications.

My remaining comments will be devoted to future research programs. As previously stated, the ALMC is the operating organization primarily concerned with the devlopment of Army-wide type logistical study programs. From a policy viewpoint, the studies will be oriented toward future wars and emphasis will be placed on those studies which offer a high probability of practical applications. The tentative FY 60 program consists of the following areas:

a. Continued development of logistics gaming capability. This capability is necessary as a training device, as a means for testing logistic plans, and as a means for evaluating alternative systems for maintenance and supply.

b. Continued efforts in logistics economics. These studies include the fundamental analysis needed to provide policy guidance in methods, procedures and trends in the basic functional areas of logistics, i.e., inventory moderization, economic inventory policy, economics of equipment maintenance and overhaul, plant and production equipment obsolescence, provisioning policy, and effects of financial controls.

c. Organization and systems analysis. These studies include elements of supply responsiveness, i.e. optimum CONUS logistics complex, Theater Army supply and maintenance organizations, applications of ADPS, logistics operations in support of limited war, missile supply and support systems, and strategic lift.

d. Development of realistic operational and administrative logistical data. These studies consist of revision to FM 101-10 "Staff Officers Field Manual," development of maintenance factors, and development of missile supply rates.

e. Lastly, our program includes the analysis of the impact of technological developments upon logistical systems, i.e. impact of missile supply and support upon logistic concepts, and the effects of non-destructive uses of atomic energy upon logistic systems.

In summary, the purposes of my discussion with you have been 1) to impress upon you the importance of logistics research and 2) to provide insight into the DA logistical research organizations, policies, programs and problem areas. I am convinced of the importance of logistics research and the need to devote greater portions of our resources to the program objective areas which I have outlined.. I hope that those of you concerned with sponsoring research or conducting research, will give more thought and attention to the problems of communication between the operators and researchers, and to the translation of study results into practical applications.

FUTURE OPERATIONS RESEARCH PROBLEMS IN ORDNANCE LOGISTICS

Herbert P. Galliher
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When I was asked to talk originally, it was on the subject of future Operations Research problems in Ordnance, and that was too much for me so I suggested the word logistics be added, but Colonel Leist has now described logistics in such a way that the subject is still much too large for me. In fact it seemed to me that the way he has described it is that it is not even a matter for Ordnance at all, or even for the Army, and this comes out not really as an attempted bit of humor but merely the statement of the facts. I also feel a little bit like a person might feel if the situation were as follows: A group of people undertook to cut down a mountain with, we'll say shovels. After a few initial spadefulls an invited speaker was brought in to discuss the subject of the future work in cutting down the mountain. However, just before him there appeared on the program a representative of the boss who said what the future work was going to be. In a similar fashion it seems to me that Colonel Leist gave a good coverage of future work in this area.

Now, I am not a member of the Ordnance corps, although we have in our work at MIT been very generously supported by the Ordnance, but in reference to what I have to say now I would like to feel that I am completely independent, that I can say what I want to say with impunity. If I feel things are not being done right, I am going to say so, I am going to try to, and see what we can turn up of value to us in our thinking. I shall think in this way, and try to get you to share this attitude with me for a little while this afternoon. It seems to me, incidentally, that it would be appropriate in the last session of this conference for me not to give a talk, but as soon as possible for us to really address ourselves, perhaps as a group, to an open discussion of this question which is the subject of the hour. In order to try and accomplish that, I am going to say some things which I hope will provoke discussion.

Now, let me take a narrow aspect of logistics in Ordnance first off, and think of the matter of supply. Actually since the bulk of our work for Ordnance has been done in connection with secondary items, and the study of the supply of these, particularly at the national level, you are being warned that there may be a bias in my attempting to talk about supply in any general terms. But if we look at the Ordnance supply system as a whole, it seems to me that it has one function, namely <u>to link the manufacturer or the designer</u> - the people who can make things - <u>with the demand</u>. The latter includes the people who need them and the kind of organization for combat or for threatened combat in peace - time that we need. But the job of Ordnance in its supply function is to correlate the agency that supplies and the agency that uses.

The entire design of Ordnance both as to people, positions, jobs, classification, the amount of effort as to depots, where they should be, whether they should be, as to what we should stock there, whether we should stock or not, as to posts, camps, and stations, whether there should be stocking points, or not, etc., all are part of the picture. Of all the basic questions of supply (and there are usually about four dominant questions - what items to stock, where to stock them, when to stock them, and how much). The answer to

all these questions is an answer to the question of how Ordnance should operate. We have only to measure the operation in terms of its performing the fundamental function, namely, linking the supplier with the user. In the end, all our present establishments might go by the board if we found that an alternative organization of the system better performed this function of linking the supplier to the user.

This problem is just obvious every time you turn around, the difficulty of doing this and the need of doing it. We even have the problem in linking supplier and user right in our research work. We cannot keep track of the research projects that are going on, and who is doing what on what project, or what has been done, or what the findings were. We cannot even coordinate this information in our own research work. It is easy to see that a problem of the size of supply throughout Ordnance which in spare parts alone, I guess, is nearly half a billion dollars worth of issues a year is really a major problem, and therefore must be a big problem of coordination. As a matter of fact to the extent that we failed to perform the right linkage, we are failing to perform the function as well as we might. One of the main aspects is communication, how can we communicate faster. Thus we cannot solve the logistics problem until we can communicate faster and better, until we can store information, more of it in retrievable form - quickly retrievable.

These general considerations are not merely general but come up specifically in any area in which you attempt to work. Let me illustrate this with respect to the matter of spare parts supply on which we have done some work, not only us but, of course, a lot of other people. If you look at the supply system as a whole, you have a map of mainly the United States, at least as far as source of supply goes, and perhaps I can do this by drawing a sketch on the board. If this is the map of the United States, and if we have our manufacturer or potential manufacturer located here, there, and everywhere, we have also our users and they are located here, there, and everywhere, not only on this map but on the map of the world. Now the national supply picture and the structure of it so far is that we divide out items into commodity groups. We divide them into six - seven now, I guess. We have also warehouses or major depots, and thereby we have really divided the map into a number of depot areas. In each area we have at least a major depot and customers, who are pretty well scattered. Items are routed, however, through posts, camps, and stations, or direct support units of which there are several hundred, and these in turn are assigned to a depot for normal supply of items. The depot here supplies demand if it has the stock. If it doesn't, the demand is referred to the system of all the depots, and may be supplied from one of the other depots, or it may be that the demand waits until stock on the way from the manufacturer - a particular manufacturer for this stock - is being allocated to all the depots.

That is the geographical picture. Now the picture in time is basically one of intermittent replinishment of the system. That is to say, every so often we have a batch of stock manufactured by the manufacturer. Once in a while we give him an order, be it every six months or every year or with some other variation. In a few cases we give standby orders to replinish this system on a continuing basis, but essentially the time picture is one of movements at some point like this, and on occasion a spread like that.

Meantime, lots of requests within the system requiring transfers of stock back and forth between units come along. If you try to draw a very simple picture of the operation as a whole, this is the sort of picture that you get.

Why should we draw a simple picture? Because this is one very important problem area that we ought to keep working on. The most important area, it seems to me, is the system area, the large area, the operation of the entire business. There are obvious reasons why that is true. In the first place the business is pretty big. It is easy for it to get sluggish. It's a difficult thing to work on and, therfore, any work on it is apt to pay off in a big way. Second, the risk of not working on it is very great, but in order to work on a big system we have to have a simple model for it, and so we have to try to get a picture of it. Once we get a picture like this we can ask questions, obviously, lots of questions.

Let's list some. How should the manufacturer be selected? When should he be selected? Well, we have lots of Army regulations and lots of pressure from Congress which dictate the way in which manufacturers are selected. There are all sorts of bases on which selections are made. Cost is an element; integrity is another; political pressure is another; national economy is another. All of these criteria dictate selection. How is each selection made? Well, there are various review boards at which these considerations are taken into account. How do the manufacturers know that a selection is being made? Well, there is a procedure for advertising in some cases, for not advertising in others. I dare say a good many manufacturers could make things if they knew that there was an opportunity to make them. Now, then as we come into the system itself, we see this system of depots, and questions arise. How many depots should we have? Where should we stock these items? What items should we stock? This is a terribly messy question. Nobody seems to have resolved this at all. We have all kinds of stockage lists. A most certain thing about a stockage list is that any item that is on it now has a chance of not being on it three months from now or six months from now. They are very unstable lists. When should we stock? How should we move our stuff around? How much should we stock at posts, camps, and stations? These are all really basic questions to which we have very few answers so far.

But perhaps we have a few. As a result of some work we have done, we have got a picture for ourselves of the operation of the depots. We've got a mathematical picture of it and we have used it to characterize the depots under these frequent replinishment schemes. We know what will happen. We've got, as a matter of fact, a simulation of the depot system that we constructed on the 704 computer, which will take any given set of secondary items, and generate by random numbers each typical demand that will occur in a year at any depot, will handle it according to certain rules, will cause inter-depot shipments, will cause items to be manufactured, etc. Incidentally, the simulation will tabulate a great deal that happens, how much waiting was incurred by the people who ask for items, how often the demand was met at the initial source, how often at some other source, how much was shipped back and forth between manufacturers. The simulation will do all this and give us a year's experience in one second on the 704. Well, from this we got a picture of just what will happen under this kind of operation which satisfies us for the moment. Consequently, I would like to look away,

that is, either down towards posts, camps, and stations, or up towards manufacturers. Perhaps our next model should be a model of the manufacturers and of how many manufacturers are needed, and how they should be located. If so, we must put in the model the constraints the politicians would like to have satisfied, because if we can't, we'll never sell it to the politicians. Thus if we look up, we observe the problem of locating manufacturers.

By the way, if we propose to do a model of this, we have in mind ultimately a data processing system which will have stored in it, and using in its operation, everything that we use in the model of the manufacturer. Whatever its capacity is, certainly it won't be easy to describe the capacity of a manufacturer in terms of a small number of numbers, so we shall have to do some extremely clever work in order to characterize the capacity of manufacturers against a potential stock list of somewhere between 25,000 and 150,000 or more items depending on the frequency with which they are demanded. At any rate, we'll have some job of doing that. But ultimately this is the kind of linkage that we would like to take advantage of, just as I would like to be able to know right now who is doing what work on this problem anywhere in the United States. If I could have the answer to that question in a hurry, I might do my work entirely differently. We might save a year's time in what we did, but I can't get the answer to that question very easily, and so I might just as well go ahead and solve the problem for myself. A lot of the time that's what happens when you buy something from the manufacturer. You say, sure, if we advertise this thing widely enough we might pick up another 10 per cent in cost, but we just don't do it. It costs us too much to advertise. We might reduce costs, also, if we had a very good formula, objective formula. This might cut out lots of the complaints from our politicians. Well, here is an area into which we ought to work, it seems to me.

Now we might go down the other way towards the posts, camps, and stations. Mr. Pear, who is here from the Signal Corp, was saying earlier in the day to me that this is an important area. One of the big questions in this area is how posts, camps, and stations should be supplied. Should they have a port which is resupplied or should it be a stocking point? Where should we resupply from? Let's not talk just about supply on one hand and maintenance on the other because these are both part of supply function. We don't care how we get this part, or how we get it into working condition, whether it is by supply or manufacturer or by maintenance. So here then we may ask ourselves what are other alternatives of operation which might be more profitable and which would be feasible. Feasibility, incidentally, rests a lot nowadays on the use of computers, and it ought to because our ability in this area is increasing very rapidly.

I would like to sketch very briefly to you a study that we are making now on the use of computers in one area of this operation. This will be a controversial subject, I know, so we will have some fun there. As a result of our inventory studies, we found in our simulations that what people were saying was true about the delay between the time that a customer at a post, camp or station asks for an item and the time that the item was shipped from some depot that had it. Quite often it would be the same level depot,

but about thirty per cent of the time it's not, which is a high fraction. That delay is pretty substantial. In fact if the item is not shipped from this depot, the delay is between thirty and forty-five days, between the time he asks for it and the time the shipment was made.

Now if you go looking into this delay, what you go looking into is the entire data processing operation from the time that he put in his demand. Well, at present it is a manual operation. (I suppose a good many of you are much more familiar with it than I am,) in which he periodically sends in a list of items that he wants. This is called a hard-copy requisition, and then this is broken down into individual requests for individual items. These are put through files at the depot, sorting, editing, seeing if his whole order can be made up, etc. If his whole order can't be made up, delays occur. In particular it generally takes about three days, or it did take about three days for that requisition to go through that operation right there, even if the stock was there.

Now if the stock was not there, this is what would happen. And notice, as I say, something like thirty per cent of the time the demand was being filled from somewhere else. (As a matter of fact that might be a good way of operating in itself if you visualize a combat operation, it might very well be true that more than thirty per cent of the time the demand was not filled from the primary source to which this unit was assigned. So you simply cannot say, 'that percentage is too high'. It might profitably be higher than that.) At any rate, what happens now, in the thirty per cent of the time, is that there are several modes of operation. If this is a request from overseas, from the European Theatre, say, it would get pretty fast handling, maybe as fast as two days. If it is not from overseas, in fact if it is a routine Army continental request, it will probably get about a twenty-five day delay, in which a lot of things have gone on in the meantime. The depot doesn't have the stock so he goes to the Supply Control Point and says we don't have the stock - we have the demand. The Supply Control Point looks at the stocks at the other depots. They say so and so has the stock, try him. So he tries so and so. But in the meantime, some days have gone by, and by the time our man tries, so and so no longer has the stock.

You see, this typically happens when everybody is short of stock toward the end of some procurement cycle. Thus you have the problem that your information is never up-to-date, and the action taken on it is action taken on something out of date, (and sometimes these delays don't ever converge!) Well, these delays are pretty considerable.

Now, consider one other little matter. Suppose this were war time, and suppose we were operating under such an organization in a Field Army Theatre. What kind of delays could we tolerate? I've made my own calculations on this as a result of participating in some studies involving this kind of operation. It turns out that the data processing delay, the tolerable delay, in such operations, is extremely short. Now, if it is short and if moreover we are having substantial delays now, what can we do about it?

In our work at MIT we are studying several schemes of computer networks to do this. Let me sketch one just very briefly, which is not necessarily one that I am recommending. But it will give us a feeling for the problem. According to some of our estimates if you had a 709 computer located somewhere-it might not always be the same 709 from hour to hour - (we could, if we had a telephone type network linking every post, camp, and station directly to that unit, and linking this unit to every depot), then roughly the following would happen. This computer would have to handle about two demands a second under a pretty heavy load on a sixteen hour a day basis, five days a week. It would have a capacity of upwards of ten demands per second during that same period of time. The utilization of the telephone lines would amount to somewhere between two, say, and ten telephone lines full time utilization. (We could not make full utilization of the telephone lines.) One thing: there would not be a punch card in the system. We are sure that if we put punch cards in the system, somebody would get their hands on them and interrupt the speed. So this will be completely electrical. The requests put in here will be put in electrically on something better than the transreceiver. This type of operation would be on to tape right now.

This computer point would also be connected, by the way, to the Supply Control Points. In fact there are about ten basic types of information that would pass over such a network, and a demand is only one of them. The others are induced messages. You have instructions to ship; you have reports of shipment. You have warehouse refusals. You have requests for content of file. You have an instruction from the supply Control Point to change the parameters. What is this computer doing, by the way? Well, it could be managing these items according to the rules used in our simulation. They are not perfect rules, but they don't seem to be too bad, at least to handle a good part of the business. What kind of response could we get on this? This is very interesting. If we had to handle 150,000 items, that the stock fund lists, it turns out that it would take us a number of hours because each time a demand comes in we have to pick out the files on this item unless we batched items. If we batched items, we might cut that time down. But let's say that it might be four hours, just as a figure, between the time that the demand was entered here on the system and the time that a shipping instruction was printed out at a depot, that, according to the file here, had the stock. However, suppose we only had to handle 25,000 items. Why I picked 25,000 is that 25,000 represents a good portion of the active items that generate most of the demands. Then we can respond in a matter of seconds. Very interesting.

Now, why do I suggest this type of unit? Not because we are recommending any such unit. Why do I suggest this type of network? We're not recommending <u>it</u>. We're going on and study some others. What are other possibilities? Other possibilities are to add less computer capacity than this, but scatter it at the depots and at other points, and to have the routing of traffic be different. What do we have to explore here? What we have to explore are the <u>alternatives</u> of operation, and for each alternative we have to compute the operating requirements or specifications. That is, what the network would have to do, how much traffic of each sort it would have to handle and at what rates, what kind of congestion we could expect from overloads. Compute these requirements for each one of these alternative network designs, and simply compare their performance in terms of response. When we've done that, what we've got is something like a cost versus performance

curve for various types of network. From hereon, in my opinion, we can leave it up to the Commanding Officer of this outfit to make the decision.

Well, here is an area in which we are working ourselves. What I'm saying is that here is a system that is operating as a whole in which there is something that obviously needs to be done. Speed it up. The quicker we do it, the better. Actually the major part of this study, it seems to me, is not even an Operations Research study. However, we are going to compute the congestion factors because there are serious problems that will occur here, such as queuing problems. I think the main job to be done here is to get some such system installed just about as quickly as possible. Of course, the fact that we can't utilize all those telephone lines in this operation means somebody else can utilize them - somebody in the other tech services. We have other kinds of messages coming over this network. Now the Signal Corps has the job of communication in the Army, but we have our requirements, too - and the other tech services have their requirements. Our problem here is to get something done quickly. If this isn't done, why the Operation Research work that was done will be kind of a failure. And there will be another weak report to Congress on what has been accomplished by Operation Research in Ordnance.

Now I would like to turn briefly to another problem about future work in Operations Research, and this has to do with the problem of measurement. Operations Research began, I think, or so people claim, to help administrators make decisions according to some objective rules. We can draw you a quantative model of an operation. We can find the optimum solution, and the net result of what was done is really this. First, they restated the problem, and redescribed the operating problem in quantitative terms. Then the optimization part says that if the value of the cost is this, and if the value of the demand is this, and if the value of the need is this, then this is the optimal strategy.

Now there are two different things here that have been done. The first part was to state the problem in quantitative form. The second is to try to apply it so as to optimize something in any given situation. If that was all that the Operation Researcher should have to do, his job is done. If it so happens that nobody can measure the values of all those things, that is not to be charged against him.

Well, I don't take that viewpoint. Frankly, if somebody can show that I have given a solution to this form and he can't measure the value of X, the value of Y, the value of Z, he's really saying well, I've been talking to myself all the time. I've got a completely hypothetical theoretical problem (and this is <u>very</u> basic research because it has no potential application at all and that fits the definition code!) If it's applied research then it must have been that I thought he could measure it. And in fact, I think our biggest problem, the big problem nowadays is, how to find the cost of procurement, and the cost of inventory, the value of being out of stock. What are these values? How do we get them? If we don't measure them, then the application is not going to be made. The research is not going to be implemented, and it will die, says Colonel Leist, and I am sure he is right. All right, I think we must take on the challenge of measurement. For example, we must take on the challenge of forecasting demand, because our

models for inventory control require that we forecast the demand. Now demand forecasting is, afterall, a tough problem. Most Operations Research people, the mathematicians in particular, will not take on this job. They are 'hypothetical' people. They only grind out solutions of the formula 'if A, then B'. It is the engineers who supply the value of A.

Our Operations Research approach, it seems to me, has got at this point to rescue itself, to take on an engineering approach and measure these things. What are the measures and how can we measure them easily? If we want to compute the cost of carrying something in inventory in various depots, clearly we can't take every item on the list and go out and cost all those things. How do we cost them? Oh, we watch them for a year and we report the cost. We index these against other things, and then we say use that as the cost, use the index. Well, we spent an awful lot of work on it and besides that we've now still got to predict the index. So we're back to the same old problem again.

I think, in other words, it's somewhat less than what you've been asked to do if you turn out a theoretical solution. (Unless your job is perhaps what ours is at MIT, which is to turn out theoretical solutions!) So there are really levels of Operations Research. One level, and let me just dwell on that point a moment, one level is modeling problems as a whole. I would like, at this point, make the point that the best Operations Research is going to be done in Ordnance by Ordnance people, by you people who are already where you are. It is not going to be done by people who come in to do it for you. It is not going to be done by your Commanding Officer nor your superiors; it is going to be done by you. What problems will you solve? All of them. The ones about the system as a whole - yes. Is that your responsibility? It doesn't make any difference; if you want to see a job get done, do a job on it. Present it at the next Operation Research conference. (Do it nights!)

Now, as far as future problems go, there are obviously lots of applications that are going to be made that I couldn't begin to predict what is going to happen. We've had a good list of them today and yesterday. One that I've wondered about is the scheduling of ammunition production done with the help of some mathematical formula. This is nearly a (linear) transportation type of problem. The last answer I had was that, that method has not yet been applied to this problem. I suspect the people who make the decision as to the schedule are in the Pentagon. If you're not in the Pentagon, then somebody's problem is how to get your solution to the attention of the people in the Pentagon, because you are their talent. Linear programming applied to a foundary cost operation was mentioned this morning. There must be lots of others like that. Incidentally, the question was raised then about the use of this by other foundries. I am sure that a lot of you know that linear programming has been used by the oil companies in a very similar type of problem, namely putting in mixtures of crude and getting out product streams. This is a matter of routine in a number of oil refineries. There is another nice area, demand forecasting. Still another nice area is product characterization. That's a tricky one and I believe no models exist. I would like to see people, who know products and who also know some Operation Research, work on it. For example, how do you characterize a product for purposes of storing its file in a data processing system? Well, you put a stock control number in.

That's an identification but not a connotation. Can you gain anything from the connotation? Can we build a product characterization which will at least do things like accounting for multiple applications for substitute ability? There are lots of points at which our descriptions of things should be a lot more indeixical than symbolic. There is a great tendency in government to use symbolic descriptions of things so there is no similarity between the name of a thing and the nature of a thing.

Well, I've made just a few suggestions here. Perhaps one other thing ought to be mentioned and that is the value of information. It seems to me that information is the biggest thing that we need and that it has a greater value than we think. Let me give you an example. Suppose that a man A makes something for a man B. Now the reason he makes it is so B can use it to make a project. Now let's suppose A invests X units of effort, and that if B knows that he's got it, he can go and get it and operate on it and get a return of Y. A invested 50 dollars in it, and B picks it up and adds something to it and he gets 60 dollars for it, so 10 dollars was made. Now let's just suppose that B didn't know that A made it. A did make it for B's use but B didn't know it. What's the net return on that? Well, we spent 50 dollars and we get nothing back. In the other case we spent 50 dollars and we got 60 back. What was the information worth? Namely the information that A had made it, for B's purposes. Well, somewhere up to 60 dollars - very interesting.

I've tried to say some things which would stimulate some discussion but maybe I have not- in which case the meeting will be that much shorter. And if I have, perhaps now would be the time to try to take up any questions you may have.*

*The discussion period is not recorded here since the questions from the floor did not come through on the tape recording.

ATTENDANCE LIST*

Name	Agency Represented
Bruce Daniel Barnett	Picatinny Arsenal, Dover, New Jersey
Thomas J. Bartlett	USARADCOM, Ent AFB, Colo. Springs, Colorado
John D. Beckmann	USA Sig Res & Dev Lab, Ft. Monmouth, New Jersey
Ralph E. Brown	Frankford Arsenal, Phila, Pennsylvania
Hans Fritz Bueckner	Math Res Cen, Wisc ARF, Madison, Wisconsin
Raymond Herbert Burros	USCONARC, Ft. Monroe, Virginia
Richard W. Cotton	Hq. OWC, Rock Island, Illinois
Dr. John W. Coy	WSMR, New Mexico
Francis George Dressel	Ofc of Ord Res, Box CM, Duke Sta, Durham, N. C.
Richard James Duffin	Duke University, Durham, N. C.
John Joseph Dunne	Ballistic Research Labs APG, Maryland
Joseph P. Fearey	JPL, Pasadena, Calif.
John Coke Flannagan	USCONARC, Ft. Monroe, Va.
George Claggett Francis	Aberdeen Prvg. Grd, Maryland
Herbert Parrish Galliher	Mass Inst of Tech., Cambridge, Mass.
Gideon Isaiah Gartner	Rodman Lab, Watertown Ars, Watertown, Mass.

*This list does not include those individuals from the host Arsenal that attended the conference.

Name	Agency Represented
Major Bertram K. Gorwitz	OCR&D, Wash, D. C.
David Meyer Herzmark	WSMR, New Mexico
Joseph Elbert Hoagbin	Opns Res Dept, Willow Run Labs, Ann Arbor, Michigan
Frank Joseph John	Watervliet Arsenal, New York
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Alfred S. Kress	Detroit Arsenal, Centerline, Michigan
Raymond N. Kreucher	Detroit Arsenal, Centerline, Michigan
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Gayle W. McElrath	Univ. of Minnesota, Dept. of Mech. Engr.
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Name	Agency Represented
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